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A CHARACTERIZATION OF MODULAR LATTICES

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1. Introduction. A binary algebra $(L,+,\cdot)$ is said to be a *lattice* if it satisfies the following identities:

$$
\begin{array}{lll}
1) & x+x=x\,, & x\cdot x=x\,,\\
2) & x+y=y+x\,, & x\cdot y=y\cdot x\,,\\
3) & (x+y)+z=x+(y+z)\,, & (x\cdot y)\cdot z=x\cdot(y\cdot z)\,,\\
4) & (x+y)\cdot y=y & x\cdot y+y=y\,.
\end{array}
$$

(In the sequel we shall write xy instead of $x\cdot y$.) A lattice $(L,+,\cdot)$ is *modular* if the identity $x(xy+z)=xy+xz$ holds in $(L,+,\cdot)$.

The main purpose of this paper is to prove the following:

THEOREM 1.1. *Let* $(L,+,\cdot)$ *be a commutative binary algebra in which the following identities hold*: $(x+y)y=y$, $x+x=x$. *Then* $(L,+,\cdot)$ *is a nondistributive modular lattice if and only if* $p_3(L,+,\cdot)=19$.

Recall that $p_n(A)$ denotes the number of all essentially n-ary polynomials over A, i.e., polynomials depending on all their variables. For this and all other undefined concepts used here we refer to [10] (see also [9]).

In his survey of equational logic, Taylor ([13], p. 41) poses a general problem of whether the numbers $p_n(A)$ characterize (to some extent and perhaps in special circumstances) the algebra A. Our result can be treated as a contribution to this problem.

An algebra (A,F) is called *idempotent* (*symmetric*) if every $f\in F$ is idempotent (symmetric). A symmetric binary algebra is called *commutative*. At the Klagenfurt Conference on Universal Algebra (June, 1982) we announced the following (see also [3]).

THEOREM 1.2. *Let* $(B,+,\cdot)$ *be a bisemilattice. Then* $(B,+,\cdot)$ *is a nondistributive modular lattice if and only if* $p_3(B,+,\cdot)=19$.

The proof of this theorem appeared in [5] (cf. [11]). At the same conference during the Problem Session we stated the following:

Conjecture 1.3. *Let* $(A, +, \cdot)$ *be a commutative idempotent binary algebra with different operations* $+$ *and* $\cdot$. *Then* $(A, +, \cdot)$ *is a nondistributive modular lattice if and only if* $p_3(A, +, \cdot) = 19$.

So, Theorem 1.1 can also be treated as a step towards the proof of this conjecture.

An algebra $(A, \{f_t\}_{i\in T})$ is said to be *proper* if the mapping $t \to f_t$ is one-to-one and every nonnullary f_t depends on all its variables. Let $f = f(x_1, \ldots, x_n)$ be a function on a set A. Then we denote by $G(f)$ the *symmetry group* of f, i.e., the set of all permutations $\sigma \in S_n$ (where S_n denotes the symmetry group of n letters) such that $f = f^\sigma$, where $f^\sigma(x_1, \ldots, x_n) = f(x_{\sigma 1}, \ldots, x_{\sigma n})$ for all $x_1, \ldots, x_n \in A$ (see [10]). A function $f = f(x_1, \ldots, x_n)$ is called *symmetric* if $f = f^\sigma$ for all $\sigma \in S_n$, and *idempotent* if $f(x, \ldots, x) = x$ for all $x \in A$.

Recall that a *bisemilattice* (see Theorem 1.2) is a commutative idempotent binary algebra $(B, +, \cdot)$ such that both $+$ and $\cdot$ are associative, i.e., both reducts $(B, +)$ and $(B, \cdot)$ are semilattices.

To prove Theorem 1.1 we need several lemmas.

2. Binary idempotent algebras. Let $(A, +, \cdot)$ be a proper idempotent binary algebra such that $(A, +)$ is commutative. Let

$$s(x, y, z) = (x + y) + z\,, \qquad \widehat{s}(x, y, z) = (xy)z\,,$$
$$f(x, y, z) = (x + y)z\,, \qquad \widehat{f}(x, y, z) = xy + z\,,$$

and if additionally $(A, \circ)$ is a proper noncommutative idempotent groupoid, then let also

$$q_1(x, y, z) = (x + y) \circ z\,, \qquad q_2(x, y, z) = z \circ (x + y)\,.$$

Similarly to [6] we get

Lemma 2.1. *If* $(A, +, \cdot)$ *is a proper idempotent binary algebra such that* $(A, +)$ *is commutative,then* s, $\widehat{s}$, f, $\widehat{f}$ *are essentially ternary and pairwise distinct. If, additionally,* $(A, \circ)$ *is a proper noncommutative groupoid, then* q_1, q_2 *are essentially ternary and the polynomials* s, $\widehat{s}$, f, $\widehat{f}$, q_1, q_2 *are pairwise distinct.*

Lemma 2.2 (cf. [7]). *If* $(A, +, \cdot)$ *is a proper commutative idempotent binary algebra satisfying* $(x + y)z = (x + z)y$, *then* $(A, +, \cdot)$ *is polynomially infinite, i.e.,* $p_n(A, +, \cdot)$ *is infinite for all* $n \geq 2$. (The dual version of this lemma is also true.)

Lemma 2.3. *If an algebra* A *contains* 3 *distinct commutative idempotent binary operations, then* $p_3(A) \geq 21$.

P r o o f. Examining the symmetry groups of the polynomials $(x+y)+z$, $(xy)z$, $(x\circ y)\circ z$, $(x+y)z$, $xy+z$, $(x+y)\circ z$, $(x\circ y)+z$, $(xy)\circ z$ and $(x\circ y)z$ and using Lemmas 2.1 and 2.2 we get our assertion.

LEMMA 2.4. *If $(A,+,\cdot)$ is a proper commutative idempotent binary algebra such that either $(A,+)$ or $(A,\circ)$ is cancellative, then $(A,+,\cdot)$ contains at least three essentially binary commutative idempotent polynomials.*

P r o o f. Assume that $(A,+)$ is cancellative. Then the polynomials $x+y$, xy, $(x+y)+(xy)$ are essentially binary and pairwise distinct, because e.g. if $x+y=(x+y)+xy$, then $x+y=(x+y)+(x+y)=(x+y)+xy$ gives $x+y=xy$.

As a corollary from Theorem 1 of [1] and the last two lemmas we get

LEMMA 2.5. *If $(A,+,\cdot)$ is a proper commutative idempotent binary algebra such that $p_3(A,+,\cdot)=19$, then both polynomials $x+2y$ and xy^2 are essentially binary.*

Here xy^k denotes $(\dots(xy)\dots y)y$ (y appearing k times), and we use $x+ky$ in the additive case, respectively.

Recall that a commutative idempotent groupoid $(G,\cdot)$ satisfying $xy=xy^2$ is called a *near-semilattice* (cf. [4]).

A groupoid $(G,\cdot)$ is *distributive* if it satisfies $(xy)z=(xz)(yz)$ and $z(xy)=(zx)(zy)$.

A groupoid $(G,\cdot)$ is *medial* if it satisfies the medial law: $(xy)(uv)=(xu)(yv)$.

LEMMA 2.6. (cf. [2]). *Let $(A,+)$ be a commutative idempotent groupoid. Then the following are equivalent*:

(i) *$(A,+)$ is a semilattice.*
(ii) *The polynomial $d(x,y,z)=(x+z)+(y+z)$ is symmetric.*
(iii) *$(A,+)$ is a distributive (medial) groupoid satisfying $x+2y=y+2x$.*

LEMMA 2.7. *If $(A,+,\cdot)$ is a proper idempotent binary algebra such that $(A,+)$ is commutative and $(x+y)z=(x+z)y$, then the polynomial $x\circ y=x+2y$ is essentially binary and noncommutative. Moreover, there exist such algebras with $(A,\circ)$ noncommutative.*

P r o o f. First we give an example. Let $(A,\oplus)$ be an abelian group of exponent 5. We put $x+y=3x\oplus 3y$ and $xy=4x\oplus 2y$. Then $(A,+,\cdot)$ is the required algebra (note that this algebra satisfies $x\circ y=xy$ and is not polynomially infinite, comp. with Lemma 2.2).

Assume now that $(x+y)z=(x+z)y$. Then $x+y=(x+y)(x+y)=((x+y)+y)x=(x\circ y)x$, thus $x\circ y$ is essentially binary. Assume that $x\circ y$ is commutative. If in addition $\cdot$ is commutative, then $x+y=(x\circ y)x=(y\circ x)x=((y+x)+x)x=x(x+y)=(x+y)x=xy$, a contradiction. If $\cdot$

is noncommutative, then $xy = (x+x)y = (y+x)x = ((x+y)+(x+y))x = ((y+x)+x)(x+y) = (y \circ x)(x+y) = (x \circ y)(y+x) = yx$, a contradiction. The proof is complete.

LEMMA 2.8. *If* $(A,+)$ *is a nonassociative commutative idempotent groupoid*, $x \circ y = x + 2y$ *and* $(A,+,\circ)$ *satisfies* $(x+y) \circ z = (x+z) \circ y$, *then the polynomial* $x \circ y + z$ *is essentially ternary and its symmetry group is trivial.*

Proof. Since $(A,+)$ is proper we infer, using $(x+y) \circ z = (x+z) \circ y$, that $(A,\circ)$ is also proper. Further, $x+y \neq x \circ y$ and therefore $(A,+,\circ)$ is a proper algebra. By Lemma 2.1, $x \circ y + z$ is essentially ternary. Lemma 2.7 proves that $x \circ y$ is noncommutative (here we put $x \circ y = xy$) and hence $x \circ y + z \neq y \circ x + z$.

Assume now that $(x+y) \circ z$ is symmetric. We show that the group $G(x \circ y + z)$ is trivial. If $x \circ y + z = y \circ z + x$, then $x + y = x \circ y + y$ and hence $x \circ y = x + 2y = (x+y) \circ y + y = y \circ x + y = x \circ y + y = x + y$. Thus $x \circ y = x + y$, which contradicts Lemma 2.7.

Let now $x \circ y + z = z \circ y + x$. Then $x + y = x \circ x + y = y \circ x + x$. Putting here $x+y$ for y we get $y \circ x = y + 2x = (x+y) \circ x + x = x \circ y + x$ and hence $x \circ y + x = y + 2x$. This implies $y + 2(y+x) = (x+y) \circ y + (x+y) = y \circ x + (x+y) = (x+y) \circ x + y = x \circ y + y = x + y$. Thus $x + y = y + 2(y+x) = (x+2y) + (x+y)$. This gives $y \circ x = (x+y) \circ y = ((x+2y)+(x+y)) \circ y = (x+2y) \circ (x+2y) = x + 2y = x \circ y$ and therefore $x \circ y = y \circ x$, a contradiction.

If $x \circ y + z = x \circ z + y$, then $x + y = x \circ y + x$ and hence $x \circ y = (x+y) \circ x = (x \circ y + x) \circ x = x \circ (x \circ y)$. Thus $x \circ y + y = x \circ (x \circ y) + y = x \circ y + x \circ y = x \circ y$. Putting $x+y$ for x in $x \circ y = x \circ y + y$ we get $y \circ x = (x+y) \circ y = (x+y) \circ y + y = y \circ x + y = y \circ y + x = x + y$, which is again impossible.

Note that the dual version of the preceding lemma is also true, i.e., we have

LEMMA 2.9. *If* $(A,+)$ *is a nonassociative commutative idempotent groupoid such that* $(A,+,\circ)$, *where* $x \circ y = x + 2y$, *satisfies* $z \circ (x+y) = y \circ (x+z)$, *then the polynomial* $x \circ y + z$ *is essentially ternary and has a trivial symmetry group.*

LEMMA 2.10. *If* $(A,+)$ *is a nonassociative commutative idempotent groupoid*, *and we put* $x \circ y = x + 2y$, *then the polynomials* $(x+y) \circ z$ *and* $z \circ (x+y)$ *cannot be simultaneously symmetric.*

Proof. If both $(x+y) \circ z$ and $z \circ (x+y)$ are symmetric, then $x \circ y = (x+x) \circ y = (y+x) \circ x = (y+x) \circ (x+x) = x \circ ((y+x)+x) = x \circ (y \circ x)$. Thus $x \circ y = x \circ (y \circ x)$, and we obtain $y \circ x = x \circ (x+y) = x \circ ((x+y) \circ x) = x \circ (x \circ y)$, so $x+y=(x+y)\circ(x+y)=x\circ((x+y)+y)=x\circ(x\circ y)=y\circ x$ and we see that $x\circ y$ is commutative, thus contradicting Lemma 2.7.

LEMMA 2.11. *If $(A,+,\cdot)$ is a proper commutative idempotent binary algebra such that $x\circ y=x+2y$ is essentially binary, noncommutative and $p_3(A,+,\cdot)<21$, then the polynomials $(x+y)\circ z$, $z\circ(x+y)$, $(xy)\circ z$ and $z\circ(xy)$ are essentially ternary and pairwise distinct.*

Proof. The first fact follows from Lemma 2.1. Lemma 2.3 implies that $(x+y)+(xy)\in\{x+y,xy\}$. Assume e.g. that $z\circ(x+y)=z\circ(xy)$. Then $x+y=(x+y)\circ(xy)=(x+y)+(xy)+(xy)$ and $xy=xy\circ(x+y)=(xy+(x+y))+(x+y)$. Since $(x+y)+(xy)$ is either $x+y$ or xy we deduce that $x+y=xy$, a contradiction.

LEMMA 2.12. *If $(A,+,\cdot)$ is a proper commutative idempotent binary algebra such that $x\circ y=x+2y$ is essentially binary and noncommutative, then $p_3(A,+,\cdot)>19$.*

Proof. Assume that $p_3(A,+,\cdot)\le 19$ and consider the ternary polynomials $s=(x+y)+z$, $\widehat{s}=(xy)z$, $f=(x+y)z$, $\widehat{f}=xy+z$, $q_1=(x+y)\circ z$, $q_2=z\circ(x+y)$, $q_1'=(xy)\circ z$, $q_2'=z\circ(xy)$ and $q=x\circ y+z$. By Lemma 2.1 they are all essentially ternary. By the assumption we deduce that $+$ is nonassociative.

If $(x+y)\circ z$ is symmetric, then $\operatorname{card}G(q)=1$ by Lemma 2.8. Using Lemma 2.10 we see that $\operatorname{card}G(q_2)=2$. If f or $\widehat{f}$ is symmetric, then Lemma 2.2 shows that $p_3(A,+,\cdot)$ is infinite. Thus we may assume that $\operatorname{card}G(f)=\operatorname{card}G(\widehat{f})=2$. Considering the polynomials s, f, $\widehat{f}$, q_2, q, q_1, $\widehat{s}$ and their symmetry groups we get

$$\begin{aligned} p_3(A,+,\cdot)\ge{}& \frac{6}{\operatorname{card}G(s)}+\frac{6}{\operatorname{card}G(f)}+\frac{6}{\operatorname{card}G(\widehat{f})}\\ &+\frac{6}{\operatorname{card}G(q_2)}+\frac{6}{\operatorname{card}G(q)}+\frac{6}{\operatorname{card}G(q_1)}+\frac{6}{\operatorname{card}G(\widehat{s})}\\ \ge{}& 3+3+3+3+6+1+1=20\,, \end{aligned}$$

a contradiction.

Assume now that neither q_1 nor q_2 is symmetric and consider s, f, $\widehat{f}$, q_1, q_2, q_1' and q_2' . If $\cdot$ is nonassociative, then using Lemma 2.11 we obtain

$$\begin{aligned} p_3(A,+,\cdot)\ge{}& \frac{6}{\operatorname{card}G(s)}+\frac{6}{\operatorname{card}G(\widehat{s})}+\frac{6}{\operatorname{card}G(f)}\\ &+\frac{6}{\operatorname{card}G(\widehat{f})}+\frac{6}{\operatorname{card}G(q_1)}+\frac{6}{\operatorname{card}G(q_1')}+\frac{6}{\operatorname{card}G(q_2')}\\ \ge{}& 3+3+3+3+3+3+1+1=20\,, \end{aligned}$$

a contradiction.

If $\cdot$ is associative, then q'_1 and q'_2 are not symmetric. In fact, if e.g. q'_1 is symmetric then $xy = xy \circ xy = ((xy)y) \circ x = xy \circ x = x \circ y$, a contradiction. As above, we get $p_3(A, +, \cdot) \geq 3+1+3+3+3+3+3+3 = 22$, which is impossible. The proof is complete.

Recall that a binary algebra $(A, +, \cdot)$ is called a *bi-near-semilattice* if both groupoids $(A, +)$ and $(A, \cdot)$ are near-semilattices. Further, two algebras with the same underlying sets and the same sets of polynomials are called *polynomially equivalent*.

LEMMA 2.13. *If* $(A, +, \cdot)$ *is a proper commutative idempotent binary algebra satisfying* $p_3(A, +, \cdot) = 19$, *then* $(A, +, \cdot)$ *is either a bi-near-semilattice, or it is polynomially equivalent to a commutative idempotent groupoid* $(A, \bullet)$ *with* $p_2(A, \bullet) = 2$. *Moreover, in the second case* $(A, +, \cdot)$ *satisfies only regular identities.*

Proof. Let $(A, +, \cdot)$ be as in the assumptions. Lemma 2.5 shows that $x \circ y = x + 2y$ is essentially binary, and so is xy^2.

Assume that $x \circ y = y \circ x$. If $x \circ y \neq x + y$, then Lemma 2.3 yields $x \circ y = xy$. Using now Theorem 4 of [1] and again Lemma 2.3 we deduce that $(A, +, \cdot)$ is polynomially equivalent to the commutative idempotent groupoid $(A, +)$ with $p_2(A, +) = 2$.

Applying Lemma 2.12 (and its dual version) we deduce that $x + 2y$ and xy^2 are commutative (and clearly essentially binary). Assume now that $(A, +, \cdot)$ is not polynomially equivalent to a groupoid. Then $x + 2y = x + y$ and $xy = xy$ and therefore $(A, +, \cdot)$ is a bi-near-semilattice.

Note that if $(A, +, \cdot)$ is polynomially equivalent to a commutative groupoid with $p_2 = 2$, then the results of [4] show that $(A, +, \cdot)$ contains a subgroupoid isomorphic to $N_2 = (\{1, 2, 3, 4\}, \square)$, where

$$x \square y = \begin{cases} x & \text{if } x = y\,, \\ 1 + \max(x, y) & \text{if } x, y \leq 3 \text{ and } x \neq y\,, \\ 4 & \text{otherwise.} \end{cases}$$

It is easy to see that N_2 satisfies only regular identities (cf. [4, 12]). The proof is complete.

Since the identity $(x + y)y = y$ in Theorem 1.1 is nonregular we see that according to the last lemma we consider in the sequel bi-near-semilattices with one absorption law.

3. Bi-near-semilattices with one absorption law.

In this section we deal with bi-near-semilattices satisfying the identity $(x + y)y = y$. First we recall the following.

THEOREM 3.1 (Theorem 6 of [6]). *Let* $(L,+,\cdot)$ *be a commutative idempotent binary algebra satisfying* $(x+y)y=y$. *Then the following conditions are equivalent*:

(i) $(L,+,\cdot)$ *is a distributive lattice.*
(ii) $(L,+,\cdot)$ *satisfies* $(x+y)z=xz+yz$.
(iii) $(L,+,\cdot)$ *satisfies* $xy+z=(x+z)(y+z)$.

Note that the idempotency of $\cdot$ follows from the idempotency of $+$ and the absorption law $(x+y)y=y$.

Let now $(A,+,\cdot)$ be a proper bi-near-semilattice satisfying $(x+y)y=y$. Consider the following ternary polynomials over $(A,+,\cdot)$:

$$
\begin{aligned}
s &= s(x,y,z)=(x+y)+z, & \widehat{s} &= \widehat{s}(x,y,z)=(xy)z\,,\\
d &= d(x,y,z)=(x+z)+(y+z), & \widehat{d} &= \widehat{d}(x,y,z)=(xz)(yz)\,,\\
f &= f(x,y,z)=(x+y)z, & \widehat{f} &= \widehat{f}(x,y,z)=xy+z\,,\\
m &= m(x,y,z)=xz+yz, & \widehat{m} &= \widehat{m}(x,y,z)=(x+z)(y+z)\,.
\end{aligned}
$$

LEMMA 3.2. *If* $(A,+,\cdot)$ *is a proper commutative idempotent binary algebra then the polynomials* s, $\widehat{s}$, d, $\widehat{d}$, f, $\widehat{f}$, m *and* $\widehat{m}$ *are essentially ternary.*

Proof. Standard, see e.g. [9].

We also have

LEMMA 3.3. *Under the same assumptions, the polynomials* s, $\widehat{s}$, f *and* $\widehat{f}$ *are pairwise distinct.*

LEMMA 3.4. *Under the same assumptions,* $m\neq\widehat{m}$, m *is different from* s *and* d, *and* m *is different from* $\widehat{s}$ *and* $\widehat{d}$.

LEMMA 3.5. *Under the same assumptions, either the symmetry groups of* f *and* $\widehat{f}$ *are two-element, or the algebra* $(A,+,\cdot)$ *is polynomially infinite.*

This follows from Lemma 2.2.

LEMMA 3.6. *If* $(A,+,\cdot)$ *is a proper commutative idempotent binary algebra satisfying* $(x+y)y=y$, *then the symmetry groups of* m *and* $\widehat{m}$ *are two-element, i.e., the polynomials admit only trivial permutations of their variables.*

Proof. Assume that $(x+z)(y+z)=(x+z)(y+x)$. Then $x+y=(x+y)(y+x)=(x+y)y=y$, a contradiction. If $xz+yz$ is symmetric, then $xz+yz=xz+yx$ and hence $xy=xy+y$. This gives $(xy)y=y$, which is impossible.

LEMMA 3.7. *If* $(A,+,\cdot)$ *is a bi-near-semilattice satisfying* $(x+y)y=y$ *such that both* $+$ *and* $\cdot$ *are nonassociative, then* $p_3(A,+,\cdot)\geq 24$.

Proof. By Lemma 3.2 the polynomials s, $\widehat{s}$, d, $\widehat{d}$, f, $\widehat{f}$, m, $\widehat{m}$ are essentially ternary. Since $(A,+,\cdot)$ is not a lattice, Theorem 3.1 shows that $m \neq f$ and $\widehat{m} \neq \widehat{f}$. Using Lemma 2.6 we infer that $s \neq d$, $\widehat{s} \neq \widehat{d}$. Further, it is routine to prove that all the above polynomials are pairwise distinct.

Since $+$ and $\cdot$ are nonassociative, Lemma 2.6 shows that $\operatorname{card} G(s) = \operatorname{card} G(\widehat{s}) = \operatorname{card} G(d) = \operatorname{card} G(\widehat{d}) = 2$. By Lemma 3.6, $\operatorname{card} G(m) = \operatorname{card} G(\widehat{m}) = 2$. According to Lemma 3.5 we may assume that $\operatorname{card} G(f) = \operatorname{card} G(\widehat{f}) = 2$. This proves that $p_3(A,+,\cdot) \geq 24$, as required.

LEMMA 3.8. *If* $(A,+,\cdot)$ *is a bi-near-semilattice satisfying* $(x+y)y = y$ *with* $+$ *associative and* $\cdot$ *nonassociative* (*or vice versa*), *then* $p_3(A,+,\cdot) \geq 20$.

Proof. Consider the ternary polynomials $s = x+y+z$, $\widehat{s} = (xy)z$, $\widehat{d} = (xz)(yz)$, $f = (x+y)z$, $\widehat{f} = xy+z$, $m = xz+yz$ and $\widehat{m} = (x+z)(y+z)$. In addition, consider the essentially ternary polynomial $g = g(x,y,z) = xy+yz+zx$. It is clear that $\operatorname{card} G(s) = \operatorname{card} G(g) = 6$. If $s = g$, then $xy+y = x+y$ and hence $x+y = (x+y)+y = (x+y)y+y = y+y = y$. By Lemma 3.2 all these ternary polynomials are essentially ternary. Applying Lemmas 3.2–3.6 and Lemma 2.6 as in the preceding proof, and examining the symmetry groups of s, $\widehat{s}$, $\widehat{d}$, f, $\widehat{f}$, m, $\widehat{m}$ and g, we obtain

$$p_3(A,+,\cdot) \geq 1+3+3+3+3+3+3+1 = 20$$

(here Theorem 3.1 has also been used). The proof is complete.

4. Proof of Theorem 1.1. Recall that our aim is to prove that a (nontrivial) commutative idempotent binary algebra $(L,+,\cdot)$ satisfying $(x+y)y = y$ is a nondistributive modular lattice if and only if $p_3(L,+,\cdot) = 19$.

First, if $(L,+,\cdot)$ is a modular nondistributive lattice, then $p_3(L,+,\cdot) = 19$ (see e.g. Theorem 1.2). Assume now that $p_3(L,+,\cdot) = 19$ and $(L,+,\cdot)$ is a commutative idempotent binary algebra satisfying $(x+y)y = y$. Lemma 2.13 shows that $(L,+,\cdot)$ is a bi-near-semilattice since it satisfies a nonregular identity $(x+y)y = y$. If this bi-near-semilattice is a bisemilattice, then the assertion follows from Theorem 1.2; otherwise, it follows from Lemmas 3.7 and 3.8. The proof is complete.

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