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ON SEMIGROUPS GENERATED BY SUBELLIPTIC OPERATORS ON HOMOGENEOUS GROUPS

BY

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1. Introduction. Let L be a positive Rockland operator on a homogeneous group G (cf. e.g. [FS]). Assume that the homogeneous degree of L is $2r$, $r>0$. B. Helffer and J. Nourrigat [HN] showed that L is hypoelliptic and satisfies the following subelliptic estimates: for every left-invariant differential operator ∂ of homogeneous degree s and every positive integer N satisfying $2Nr \ge s$ there is a constant C such that

$$(1.1)\qquad \|\partial f\|_{L^2} \le C(\|L^N f\|_{L^2} + \|f\|_{L^2}) \quad \text{for } f \in C_c^\infty(G)\,.$$

Applying these facts G. B. Folland and E. M. Stein [FS] proved that the closure $\overline{L}$ of the essentially selfadjoint operator L is the infinitesimal generator of the semigroup $\{T_t\}_{t>0}$ of linear operators on $L^2(G)$ which has the form

$$(1.2)\qquad T_t f = f * p_t, \quad t>0\,, \qquad \text{where } p_t \text{ belong to the Schwartz space } \mathcal{S}(G)\,.$$

On the other hand, it was proved by A. Hulanicki and the author [DH] that if a positive Rockland operator L is a sum of even powers of left-invariant vector fields, then the kernels p_t, $t>0$, of the semigroup generated by $\overline{L}$ have the following exponential decay: for every constant $C>0$, every $t>0$, and every left-invariant differential operator ∂ on G

$$\|(\partial p_t)e^{C\tau}\|_{L^\infty} \le c(C,t,\partial) < \infty\,,$$

where τ is a Riemannian distance from the unit element.

The purpose of the present paper is an extension of this result to semigroups generated by abstract positive Rockland operators. Actually, we prove the following theorem:

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THEOREM (1.3). *For every* $C \geq 0$, *the semigroup defined by* (1.2) *is holomorphic on* $L^2(e^{C\tau(x)}\,dx)$ *in the right half-plane and the kernels* p_z, $\operatorname{Re} z > 0$, *satisfy*

$$\|(\partial p_z)e^{C\tau}\|_{L^\infty} \leq c(C, z, \partial) < \infty$$

for every left-invariant differential operator ∂ *on* G.

It seems likely that this result can be strengthened:

$$\sup_{x\in G} |(\partial p_z(x))e^{c|x|^\alpha}| < C(\partial, z) < \infty$$

for some $\alpha > 1$, where $|\cdot|$ is a homogeneous norm on G. If the generator is as in [DH] and z is real this has been proved by W. Hebisch in [He].

It is worth pointing out that the methods we present here allow one to obtain the same theorem for the semigroup generated by the convolution with the distribution φP^N, where P is the generating functional of a δ-stable semigroup of symmetric measures on a homogeneous group G with a smooth Lévy measure, $\delta \in (0, 2)$, $P^N = P * P * \ldots * P$ (Ntimes), $N > 0$, $\varphi \in C_c^\infty(G)$, $\varphi \equiv 1$ in a neighborhood of the origin. It is easy to check that the distribution P has the following form:

$$\langle P, f\rangle = \lim_{\varepsilon\to 0} \int\limits_{|x|>\varepsilon} \frac{f(0) - f(x)}{|x|^{Q+\delta}} \Omega(x)\,dx, \tag{1.4}$$

where $\Omega \in C^\infty(G \setminus \{0\})$, $\Omega \geq 0$, $\Omega \not\equiv 0$, $\Omega(x^{-1}) = \Omega(x)$, $\Omega(\delta_t x) = \Omega(x)$, Q is the homogeneous dimension of G.

For brevity we concentrate only on semigroups generated by Rockland operators. The same arguments work for semigroups associated with the distribution φP^N.

Our proof is similar in spirit to that presented in [DH]. Since distributions considered here are not supported by the origin, as was the case in [DH], we use the Taylor expansion instead of the Leibniz formula. Subelliptic estimates which have been obtained by B. Helffer and J. Nourrigat [HN] for Rockland operators, and by P. Głowacki [G] for generators of stable semigroups of measures play here a decisive role.

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2. Preliminaries. A *family of dilations* on a nilpotent Lie algebra G is a one-parameter group $\{\delta_t\}_{t>0}$ of automorphisms of G determined by

$$\delta_t \mathbf{e}_j = t^{d_j}\mathbf{e}_j,$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is a linear basis for G, and $d_1, \ldots, d_n$ are positive real numbers called the *exponents of homogeneity.* The smallest d_j is assumed to be 1.

If we regard G as a Lie group with multiplication given by the Campbell–Hausdorff formula, then the dilations δ_t are also automorphisms of the group structure of G, and the nilpotent Lie group G equipped with these dilations is said to be a *homogeneous group.*

The *homogeneous dimension* of G is the number Q defined by

$$d(\delta_t x) = t^Q \, dx \, ,$$

where dx is a right-invariant Haar measure on G.

We choose and fix a *homogeneous norm* on G, that is, a continuous nonnegative symmetric function $x \mapsto |x|$ which is, moreover, smooth on $G \setminus \{0\}$ and satisfies

$$|\delta_t x| = t|x|, \qquad |x| = 0 \text{ if and only if } x = 0 \, .$$

Let

$$X_j f(x) = \frac{d}{dt}\bigg|_{t=0} f(xt\mathbf{e}_j), \qquad Y_j f(x) = \frac{d}{dt}\bigg|_{t=0} f(t\mathbf{e}_j x)$$

be left- and right-invariant basic vector fields. If $I = (i_1, \ldots, i_n)$ is a multi-index, $i_j \in \mathbb{N} \cup \{0\}$, we set

$$X^I f = X_1^{i_1} \ldots X_n^{i_n} f, \quad Y^I f = Y_1^{i_1} \ldots Y_n^{i_n} f, \quad |I| = i_1 d_1 + \ldots + i_n d_n \, ,$$
$$\|I\| = i_1 + \ldots + i_n, \quad I! = i_1! \ldots i_n!, \quad x^I = x_1^{i_1} \ldots x_n^{i_n} \, ,$$

where $x = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n$. The number $|I|$ is called the *homogeneous length* of I.

For a real number $r \ge 0$ let $\bar{r}$ be the smallest number such that $\bar{r} > r$ and $\bar{r} = |I|$ for some multi-index I.

For a function $f \in C_c^\infty(G)$, $\; r > 0$, $\; x \in G$, define

$$f^{(x)}(y) = f(xy) - \sum_{|I| \le r} \frac{1}{I!} X^I f(x) y^I, \qquad y \in G \, . \tag{2.1}$$

THEOREM (2.2) (cf. [FS, Theorem 1.37]). *For $r, a > 0$, there are constants C, K such that for every $f \in C^\infty(G)$*

$$|f^{(x)}(y)| \le C f^{\langle r \rangle}(x) |y|^{\bar{r}} \qquad \text{for } |y| \le a \, ,$$

where $f^{\langle r \rangle}(x) = \sum_{I \in W} \sup_{|z| \le K} |X^I f(xz)|$, $\quad W = \{I : r < |I|, \; \|I\| \le [r] + 1\}$.

We say that a function f on G belongs to the *Schwartz space* $\mathcal{S}(G)$ if for every $M > 0$ the norm

$$\sup_{|I| < M \, , \; x \in G} (1 + |x|)^M |X^I f(x)|$$

is finite.

A distribution U on G is said to be a *kernel of order* $r \in \mathbb{R}$ if U coincides with a C^∞ function away from the origin, and satisfies

$$\langle U, f \circ \delta_t \rangle = t^r \langle U, f\rangle \quad \text{for } f \in C_c^\infty(G),\ t > 0\,.$$

If U is a kernel of order r then there exists a function Ω_U, homogeneous of degree 0 and smooth away from the origin, and a differential operator ∂ such that

$$(2.3)\quad \langle U, f\rangle = \partial f(0) + \lim_{\varepsilon\to 0} \int\limits_{|x|>\varepsilon} \frac{\Omega_U(x)}{|x|^{Q+r}} \Big(f(x) - \sum_{|I|<r} \frac{1}{I!} X^I f(0) x^I \Big)\, dx\,,$$

for $f \in C_c^\infty(G)$ (cf. [G, p. 560]).

A distribution T smooth away from 0, supported in a compact set and coinciding with a kernel of order r in a neighborhood of 0 will be called a *truncated kernel of order* r. Note that if T is a truncated kernel of order r, then

$$T_I = (-x)^I T$$

is a truncated kernel of order $r - |I|$.

We say that a kernel U of order $r > 0$ satisfies the *Rockland condition* if for every nontrivial irreducible unitary representation π of G the linear operator π_U is injective on the space of C^∞ vectors of π.

If a kernel U of order $r > 0$ has compact support, i.e. $\Omega_U \equiv 0$ (cf. (2.3)], then U is supported at the origin. Hence

$$(2.4)\qquad U = \sum_{|I|=r} a_I X^I\,.$$

If an operator of the form (2.4) satisfies the Rockland condition, then U is called the *Rockland operator*.

A function w on G is *submultiplicative* if

(i) w is symmetric, Borel and bounded on compact sets,
(ii) $w(x) \ge 1$, $x \in G$,
(iii) $w(xy) \le w(x)w(y)$ for all $x, y \in G$.

Let $d(x, y)$ be a fixed left-invariant Riemannian metric on G and let

$$(2.5)\qquad \tau(x) = d(x, 0)\,.$$

For a fixed nonnegative function $f_0 \in C_c^\infty(\{x : \tau(x) < 1\})$ such that $\int_G f_0(x)\, dx = 1$ define

$$(2.6)\qquad \phi(x) = e^{\tau * f_0(x)}\,.$$

LEMMA (2.7). *For every submultiplicative function w on G there exist positive numbers m and C such that*

$$w(x) \le C\phi^m(x)\,.$$

In particular, $e^{\tau(x)} \le C\phi^m(x)$ *for some* C *and* m.

Proof. See e.g. [H, Proposition 1.2 and Lemma 4.2].

LEMMA (2.8). *For every positive* m *there exists a constant* C *such that*

$$\phi^m(x^{-1}) \le C\phi^m(x), \qquad \phi^m(xy) \le C\phi^m(x)\phi^m(y)\,.$$

Moreover, *for every left-invariant differential operator* ∂ *there is a constant* $C = C(\partial, m)$ *such that*

$$|\partial\phi^m(x)| \le C\phi^m(x)\,.$$

Proof. Cf. [H].

A subset Γ of G is said to be *uniformly discrete* if for every function $\varphi \in C_c^\infty(G)$ the function $\sum_{z\in\Gamma} \lambda_z\varphi$ is bounded, where $\lambda_z\varphi(x) = \varphi(zx)$.

The following lemma is due to B. Helffer and J. Nourrigat (cf. [HN]).

LEMMA (2.9). *For every homogeneous group* G *there is a uniformly discrete subset* Γ *of* G *and a function* $\psi \in C_c^\infty(G)$ *such that*

$$\sum_{z\in\Gamma} |\psi_z(x)|^2 = 1, \quad \textit{where} \quad \psi_z(x) = \lambda_z\psi(x)\,.$$

LEMMA (2.10). *For every uniformly discrete subset* Γ *of* G *and* $\varepsilon > 0$

$$\sum_{z\in\Gamma} (1 + |z|)^{-Q-\varepsilon} < \infty\,.$$

Proof. It suffices to show that $\sum_{z\in\Gamma,\,|z|>s} |z|^{-Q-\varepsilon} < \infty$ for sufficiently large s. Let $\varphi \in C_c^\infty(G)$, $\varphi \ge 0$, $\varphi(x) = 1$ for $|x| < 1$. Then

$$\begin{aligned}
\sum_{z\in\Gamma,\ |z|>s} |z|^{-Q-\varepsilon} &\le C \sum_{z\in\Gamma,\ |z|>s} |z|^{-Q-\varepsilon} \int \varphi_z(x)\,dx \\
&\le C \sum_{z\in\Gamma,\ |z|>s} \int \varphi_z(x)|x|^{-Q-\varepsilon}\,dx \\
&\le C \int_{|x|>1} |x|^{-Q-\varepsilon}\,dx < \infty\,.
\end{aligned}$$

COROLLARY (2.11). *If* $m > 0$, *then* $\int \phi^{-m}(x)\,dx < \infty$, *where* ϕ *is defined by* (2.6). *Moreover*, *if* Γ *is a uniformly discrete subset of* G, *then* $\sum_{z\in\Gamma} \phi^{-m}(z) < \infty$.

A semigroup $\{T_t\}_{t>0}$ of bounded linear operators on a Banach space $\mathcal{X}$ is said to be *holomorphic* in the sector $\Delta_\delta = \{z \in \mathbb{C} : |\operatorname{Arg} z| < \delta\}$ if there exists a family $\{T_z\}_{z\in\Delta_\delta}$ of bounded linear operators on $\mathcal{X}$ such that

(a) $T_z = T_t$ for $z = t$ and $\Delta_\delta \ni z \mapsto T_z$ is holomorphic,
(b) $T_{z_1+z_2} = T_{z_1}T_{z_2}$ for $z_1, z_2 \in \Delta_\delta$,
(c) $\lim_{z\to 0,\, z\in\Delta_{\delta-\varepsilon}} T_z x = x$ for every $\varepsilon > 0$, $x \in \mathcal{X}$.

The *infinitesimal generator* A of the semigroup $\{T_t\}$ is defined by $\mathcal{D}(A) = \{x \in \mathcal{X} : \lim_{t\to 0} t^{-1}(x - T_t x)$ exists in $\mathcal{X}\}$, and for $x \in \mathcal{D}(A)$, $Ax = \lim_{t\to 0} t^{-1}(x - T_t x)$.

Similarly to [DH] the following theorem is the basic tool of the present paper:

THEOREM (2.12). *Let $\mathcal{H}$ and $\mathcal{V}$ be Hilbert spaces equipped with inner products $(\cdot,\cdot)_{\mathcal{H}}$, $(\cdot,\cdot)_{\mathcal{V}}$ respectively. Assume that $\mathcal{V}$ is a dense subspace of $\mathcal{H}$ such that for a constant C*

$$\|x\|_{\mathcal{H}} \le C\|x\|_{\mathcal{V}} \quad \textit{for all } x \in \mathcal{V}.$$

Let $a(u,v)$ be a bounded sesquilinear form on $\mathcal{V}$. It defines an operator $A : \mathcal{D}(A) \to \mathcal{H}$ as follows:

$$\mathcal{D}(A) = \{u \in \mathcal{V} : |a(u,v)| \le C_u\|v\|_{\mathcal{H}} \ \textit{for } v \in \mathcal{V}\}, \qquad (Au,v)_{\mathcal{H}} = a(u,v).$$

Assume that for some $\alpha,\beta > 0$

$$\alpha\|u\|_{\mathcal{V}}^2 \le \operatorname{Re} a(u,u), \qquad |\operatorname{Im} a(u,u)| \le \beta\|u\|_{\mathcal{V}}^2. \tag{2.13}$$

Then A is the infinitesimal generator of a strongly continuous semigroup of operators on $\mathcal{H}$ which is holomorphic in the sector Δ_δ, $\delta = \arctan(\alpha/\beta)$, and uniformly bounded in every proper subsector of Δ_δ.

Proof. Cf. [DH] and [P, Theorem 5.2].

3. Subelliptic estimates. Let L be a positive Rockland operator on G, homogeneous of degree $2r$, and let $E_{\bar{L}}$ be the spectral resolution for $\bar{L}$. Since L is homogeneous and symmetric, the kernels p_t of the semigroup $\{T_t\}_{t>0}$ generated by $\bar{L}$ (cf. (1.2)) are symmetric and satisfy

$$p_t(x) = t^{-Q/(2r)} p_1(\delta_{t^{-1/(2r)}} x). \tag{3.1}$$

Let $\{S_t\}_{t>0}$ be the semigroup (subordinate to $\{T_t\}$) generated by $\sqrt{\bar{L}}$, that is,

$$S_t f = \int_0^\infty e^{-t\lambda^{1/2}}\, dE_{\bar{L}}(\lambda) f = \int_0^\infty \frac{e^{-s}}{\sqrt{\pi s}} f * p_{t^2/(4s)}\, ds. \tag{3.2}$$

Obviously

$$S_t f = f * q_t, \quad \text{where} \quad q_t = \int_0^\infty \frac{e^{-s}}{\sqrt{\pi s}} p_{t^2/(4s)}\, ds. \tag{3.3}$$

It follows from (3.1) and (3.3) that $q_t \in C^\infty(G) \cap L^1(G)$, and

$$q_t(x) = t^{-Q/r} q_1(\delta_{t^{-1/r}} x). \tag{3.4}$$

The infinitesimal generator of $\{S_t\}$ on $C_{\mathrm{c}}^{\infty}(G)$ is the convolution with the distribution U defined by

$$\langle U, f\rangle = c^{-1}\int_0^{\infty} t^{-3/2}\Big(\int_G f(x)p_t(x)\,dx - f(0)\Big)\,dt\,, \tag{3.5}$$

where $c=\int_0^{\infty} t^{-3/2}(e^{-t}-1)\,dt$. (3.5) implies that U is a kernel of order r. Of course

$$\langle U * U, f\rangle = Lf(0)\,. \tag{3.6}$$

Note that

$$(\mathrm{Id}+\sqrt{L})^{-1}f = f * F\,, \tag{3.7}$$

where

$$F = \int_0^{\infty} e^{-t}q_t\,dt \in L^1(G)\,. \tag{3.8}$$

PROPOSITION (3.9). *For every kernel T of order $s>0$ and every positive integer N satisfying $Nr\ge s$ there is a constant C such that*

$$\|f*T\|_{L^2(G)} \le C(\|f*U^N\|_{L^2(G)} + \|f\|_{L^2(G)}) \quad \textit{for } f\in C_{\mathrm{c}}^{\infty}(G)\,. \tag{3.10}$$

Proof. The proof proceeds by induction on the step of G. If G is abelian, then (3.10) follows by using the Fourier transform. Assume that (3.10) holds for groups of step $< m$, and let G be a homogeneous group of step m. Let V denote the center of G. Let S be a linear complement to V which is invariant under the action of dilations. Then S can be considered as a homogeneous group isomorphic to G/V. Denote by σ the canonical homomorphism from G into S. The operator $\widetilde{L}$ defined on $C_{\mathrm{c}}^{\infty}(S)$ by

$$\widetilde{L}f = L(f\circ\sigma),\qquad f\in C_{\mathrm{c}}^{\infty}(S)\,,$$

is a positive Rockland operator on S. Moreover, the distribution $\widetilde{U}$ subordinate to the kernel $\widetilde{L}$ satisfies

$$\langle \widetilde{U}, f\rangle = \langle U, f\circ\sigma\rangle \qquad \text{for } f\in C_{\mathrm{c}}^{\infty}(S)\,.$$

Let T be a kernel of order s and let N be such that $Nr\ge s$. Then $\widetilde{T}$ defined by $\langle\widetilde{T}, f\rangle = \langle T, f\circ\sigma\rangle$, $f\in C_{\mathrm{c}}^{\infty}(S)$, is a kernel of order s on S (cf. [G, (3.26)]) and by our inductive assumption there is a constant C such that

$$\|f*\widetilde{T}\|_{L^2(S)} \le C(\|f*\widetilde{U}^N\|_{L^2(S)} + \|f\|_{L^2(S)}) \qquad \text{for } f\in C_{\mathrm{c}}^{\infty}(S)\,. \tag{3.11}$$

Of course $f*\widetilde{U}^N = \pi^0_{\check{U}^N}f$ and $f*\widetilde{T} = \pi^0_{\check{T}}f$, where π^0 is the unitary representation induced from the trivial character on V, $\langle\check{T}, f\rangle = \langle T, \check{f}\rangle$, $\check{f}(x)=f(x^{-1})$. Hence (3.11) can be written as

$$\|\pi^0_{\check{T}}f\|_{L^2(S)} \le C(\|\pi^0_{U^N}f\|_{L^2(S)} + \|f\|_{L^2(S)})\,. \tag{3.12}$$

It was shown in [G, pp. 568–571] that if U satisfies (3.12) and the kernel of $(\mathrm{Id}+U)^{-1}$ belongs to $L^1(G)$ (cf. (3.7), (3.8)), then there is another constant C such that

$$\|\pi^{\xi}_{\check{T}} f\|_{L^2(S)} \le C(\|\pi^{\xi}_{U^N} f\|_{L^2(S)} + \|f\|_{L^2(S)}) \quad \text{for } f \in C_{\mathrm{c}}^{\infty}(S),\ \xi \in V^*, \tag{3.13}$$

where π^{ξ} is the unitary representation of G induced from the character $V \ni v \mapsto e^{i\langle \xi, v\rangle}$. Decomposing the right regular representation of G into a direct integral of π^{ξ} and using (3.13), we get (3.10).

Let φ_0 be a smooth symmetric function with compact support such that $\varphi_0 = 1$ in a neighborhood of the origin. Define the truncated kernel R by

$$R = \varphi_0 U. \tag{3.14}$$

Note that there is a real symmetric function $\omega \in C_{\mathrm{c}}^{\infty}(G)$ such that

$$L = R^2 + \omega \quad \text{in the sense of distributions.} \tag{3.15}$$

From (3.9) and (2.3), we deduce the following

Corollary (3.16). *For every multi-index I with $|I| > 0$ and every $\varepsilon > 0$ there is a constant C_ε such that*

$$\|f * R_I\|_{L^2(G)} \le \varepsilon \|f * R\|_{L^2(G)} + C_\varepsilon \|f\|_{L^2(G)} \quad \textit{for } f \in C_{\mathrm{c}}^{\infty}(G). \tag{3.17}$$

Moreover, if $|I| \ge r$, then there is a constant C such that

$$\|f * R_I\|_{L^2(G)} \le C \|f\|_{L^2(G)} \quad \textit{for } f \in C_{\mathrm{c}}^{\infty}(G). \tag{3.18}$$

4. Weighted subelliptic estimates. For a fixed $m \ge 0$ put

$$\eta(x) = \phi^m(x), \tag{4.1}$$

where ϕ is defined by (2.6).

Denote by $\mathcal{H}$ the Hilbert space $L^2(G)$, and by $\mathcal{H}_\eta$ the Hilbert space $L^2(G, \eta\, dx)$, that is, $f \in \mathcal{H}_\eta$ if and only if

$$\|f\|^2_{\mathcal{H}_\eta} = \|f\|^2_\eta = \int_G |f(x)|^2 \eta(x)\, dx < \infty. \tag{4.2}$$

Our aim in this section is to prove the following theorem which is a weighted version of Corollary (3.16).

Theorem (4.3). *Let R be a truncated kernel of order $r > 0$ which satisfies* (3.17). *Then for every multi-index I with $|I| > 0$ and every $\varepsilon > 0$ there is a constant C_ε such that*

$$\|f * R_I\|^2_\eta \le \varepsilon \|f * R\|^2_\eta + C_\varepsilon \|f\|^2_\eta \quad \textit{for } f \in C_{\mathrm{c}}^{\infty}(G). \tag{4.4}$$

Moreover, if $|I| \ge r$, then there is a constant C such that

$$\|f * R_I\|^2_\eta \le C \|f\|^2_\eta \quad \textit{for } f \in C_{\mathrm{c}}^{\infty}(G). \tag{4.5}$$

P r o o f. Our proof consists of four lemmas.

We say that a linear operator B bounded on $L^2(G)$ has *compact support* if for every $a > 0$ there is a constant b such that

$$(Bf)\chi_{B(x,a)} = B(f\chi_{B(x,b)})\chi_{B(x,a)}\,,$$

where $B(x,r) = \{y : \ |x^{-1}y| < r\}$ and $\chi_{B(x,r)}$ is the indicator of the ball $B(x,r)$.

LEMMA (4.6). *If B is a bounded compactly supported linear operator on $L^2(G)$, then there is a constant C, which depends on η and the support of B, such that*

$$\|B\|_{\mathcal{H}_\eta\to\mathcal{H}_\eta} \le C\|B\|_{\mathcal{H}\to\mathcal{H}}\,.$$

P r o o f. Let ψ, Γ be as in Lemma (2.9) and let $f \in \mathcal{H}_\eta$. Using Lemma (2.8), we get

$$\|Bf\|_\eta^2 = \int\limits_G \sum_{z\in\Gamma} |Bf(x)\psi_z(x)|^2\eta(x)\,dx \le C_1\sum_{z\in\Gamma}\eta(z^{-1})\int\limits_G |Bf(x)\psi_z(x)|^2\,dx\,.$$

Since B is bounded on $L^2(G)$ and compactly supported, there is a constant $a > 0$ such that

$$\|Bf\|_\eta^2 \le C_2\sum_{z\in\Gamma}\eta(z^{-1})\int\limits_G |B(f\chi_{B(z^{-1},a)})(x)\psi_z(x)|^2\,dx\,.$$

By Lemma (2.8), we obtain

$$\begin{aligned}\|Bf\|_\eta^2 &\le C_3\|B\|^2_{\mathcal{H}\to\mathcal{H}}\sum_{z\in\Gamma}\eta(z^{-1})\int\limits_G |f(x)\chi_{B(z^{-1},a)}(x)|^2\,dx\\ &\le C_4\|B\|^2_{\mathcal{H}\to\mathcal{H}}\sum_{z\in\Gamma}\int\limits_G |f(x)\chi_{B(z^{-1},a)}(x)|^2\eta(x)\eta(x^{-1}z^{-1})\,dx\\ &\le C\|B\|^2_{\mathcal{H}\to\mathcal{H}}\|f\|^2_\eta\,.\end{aligned}$$

LEMMA (4.7). *For every truncated kernel T of order 0 there is a constant $C > 0$ such that*

$$\|f*T\|_{L^2(G)} \le C\|f\|_{L^2(G)}, \qquad f \in C_{\mathrm{c}}^\infty(G)\,.$$

P r o o f. See Goodman [Go].

R e m a r k. Note that (4.5) is now a consequence of (4.7), (4.6), (3.17), and (2.3).

LEMMA (4.8). *Let R be a truncated kernel of order r which satisfies* (3.17). *Then there is a constant C such that for every multi-index I_0 with*

$|I_0| > 0$ *and every* $\varepsilon > 0$ *there is a constant* C_ε *such that for* $f \in C_c^\infty(G)$

$$\|f * R_{I_0}\|_\eta^2 \le \varepsilon \sum_{0\le |I|<r} \|f * R_I\|_\eta^2 + C_\varepsilon \|f\|_\eta^2 + C \sum_{0<|J|\le r-|I_0|} \|f * R_{I_0+J}\|_\eta^2 .$$

Proof. We need only consider the case when $0 < |I_0| < r$ (cf. the remark following Lemma (4.7)). Let Γ and ψ be as in Lemma (2.9). Then

$$\begin{aligned}\|f * R_{I_0}\|_\eta^2 &\le C_1 \sum_{z\in\Gamma} \eta(z^{-1}) \int_G |f * R_{I_o}(x)\psi_z(x)|^2\,dx \\ &\le C_2 \sum_{z\in\Gamma} \eta(z^{-1}) \int_G |(f\psi_z) * R_{I_0}(x)|^2\,dx \\ &\quad + C_2 \sum_{z\in\Gamma} \eta(z^{-1}) \int_G \Big| \sum_{0<|J|\le r} \frac{1}{J!} X^J \psi_z(x) f * R_{I_0+J}(x) + H_z f(x)\Big|^2\,dx ,\end{aligned}$$

where $H_z f(x) = \langle (\cdot)^{I_0} R, f(x\,\cdot)\psi_z^{(x)}(\cdot)\rangle$ (cf. (2.1) for the definition of $\psi_z^{(x)}$). By (3.17), we have

$$\begin{aligned}(4.9)\quad \|f * R_{I_0}\|_\eta^2 \le{}& C_2 \sum_{z\in\Gamma} \eta(z^{-1})\varepsilon \int_G |(f\psi_z) * R(x)|^2\,dx \\ &+ C_2 \sum_{z\in\Gamma} \eta(z^{-1}) C_\varepsilon \int_G |f\psi_z(x)|^2\,dx \\ &+ C_3 \sum_{z\in\Gamma} \eta(z^{-1}) \sum_{0<|J|\le r} \int_G |X^J \psi_z(x) f * R_{I_0+J}(x)|^2\,dx \\ &+ C_3 \sum_{z\in\Gamma} \eta(z^{-1}) \int_G |H_z f(x)|^2\,dx .\end{aligned}$$

Since Γ is uniformly discrete, the first term on the right-hand side of (4.9) can be estimated by

$$\begin{aligned}&C_2 \sum_{z\in\Gamma} \eta(z^{-1})\varepsilon \int_G \Big|\psi_z(x) f * R(x) + \sum_{0<|J|\le r} \frac{1}{J!} X^J \psi_z(x) f * R_J(x) + H'_z f(x)\Big|^2 dx \\ &\quad \le C_3\varepsilon \|f * R\|_\eta^2 + C_3\varepsilon \sum_{0<|J|\le r} \|f * R_J\|_\eta^2 + C_3\varepsilon \sum_{z\in\Gamma} \eta(z^{-1}) \int_G |H'_z f(x)|^2\,dx ,\end{aligned}$$

where $H'_z f(x) = \langle R, f(x\,\cdot)\psi_z^{(x)}(\,\cdot\,)\rangle$. By (2.9) the second term on the right-hand side of (4.9) is estimated by

$$C_3 C_\varepsilon \|f\|_\eta^2 .$$

Similarly, the third term on the right-hand side of (4.9) is estimated by

$$C_4 \sum_{0<|J|\le r} \|f * R_{I_0+J}\|_\eta^2 .$$

By virtue of (4.5), we get

$$\begin{aligned}
&\|f * R_{I_0}\|_\eta^2 \\
&\quad \le C_5\varepsilon\|f * R\|_\eta^2 + \sum_{0<|J|\le r} C_5\varepsilon\|f * R_J\|_\eta^2 + C_5\sum_{z\in\Gamma}\int_G \eta(z^{-1})|H'_z f(x)|^2 dx \\
&\qquad + C_5 C_\varepsilon\|f\|_\eta^2 + C_5 \sum_{0<|J|\le r-|I_0|} \|f * R_{I_0+J}\|_\eta^2 \\
&\qquad + C_5\sum_{z\in\Gamma}\eta(z^{-1})\int_G |H_z f(x)|^2\,dx .
\end{aligned}$$

The proof of Lemma (4.8) will be completed if we show

$$\sum_{z\in\Gamma}\eta(z^{-1})\int_G (|H'_z f(x)|^2 + |H_z f(x)|^2)\,dx \le C\|f\|_\eta^2 .$$

Note that by Theorem (2.2), $\psi_z^{(x)}(y)$ is a smooth function of x, y. Moreover, for every constant $K > 0$ there is a constant $a > 0$ such that

$$\begin{gathered}
|\psi_z^{(x)}(y)| \le C_K|y|^{\bar r} \quad \text{for } |y|\le K,\ z\in\Gamma . \\
\psi_z^{(x)}(y) = 0 \quad \text{for } x\notin B(z^{-1},a),\ |y|\le K,\ z\in\Gamma .
\end{gathered}$$

Hence by (2.3) there is a constant C such that

$$\|H_z f\|_{L^2} \le C\|f\chi_{B(z^{-1},a)}\|_{L^2} \quad \text{for } z\in\Gamma,\ f\in C_{\mathrm c}^\infty(G) .$$

Consequently, by (2.8) and (2.9), we get

$$\begin{aligned}
&\sum_{z\in\Gamma}\eta(z^{-1})\int_G |H_z f(x)|^2\,dx \\
&\qquad \le C\sum_{z\in\Gamma}\eta(z^{-1})\int_G |f(x)\chi_{B(z^{-1},a)}(x)|^2\,dx \le C\|f\|_\eta^2 .
\end{aligned}$$

We proceed with H'_z analogously.

Lemma (4.10). *Let R be a truncated kernel of order $r > 0$ which satisfies* (3.17)*. Then for every multi-index I with $0 < |I| < r$ and every $\varepsilon > 0$ there exists a constant C_ε such that*

$$\begin{aligned}
(4.11)\quad &\|f * R_I\|_\eta^2 \\
&\le \varepsilon\|f * R\|_\eta^2 + \varepsilon\sum_{0<|J|<|I|}\|f * R_J\|_\eta^2 + C_\varepsilon\|f\|_\eta^2, \quad f\in C_{\mathrm c}^\infty(G) .
\end{aligned}$$

Proof. Let $\{k_1,\dots,k_m\}=\{|I| : 0<|I|<r\}$. We can assume that $r>k_1>\dots>k_m=1$. Let I be such that $|I|=k_1$. By Lemma (4.8) and (4.5) for every $\varepsilon>0$ there is C_ε such that

$$\|f*R_I\|_\eta^2 \le \varepsilon\|f*R\|_\eta^2+\varepsilon\sum_{0<|J|<k_1}\|f*R_J\|_\eta^2+\varepsilon\sum_{|J|=k_1}\|f*R_J\|_\eta^2+C_\varepsilon\|f\|_\eta^2\,.$$

Summing the above inequalities over all I with $|I|=k_1$, we conclude that for every $\varepsilon>0$ there exists a constant C_ε such that

$$\|f*R_I\|_\eta^2\le\varepsilon\|f*R\|_\eta^2+\varepsilon\sum_{0<|J|<k_1}\|f*R_J\|_\eta^2+C_\varepsilon\|f\|_\eta^2 \qquad \text{for } f\in C_c^\infty(G),\ |I|=k_1\,.$$

Assume now that (4.11) holds for $|I|=k_1,\dots,k_j$. We show that (4.11) holds for $|I|=k_{j+1}$. Let I be such that $|I|=k_{j+1}$. By virtue of Lemma (4.8) and (4.5) there is a constant C such that for every $\varepsilon>0$ there is a constant C_ε such that

$$\begin{aligned}\|f*R_I\|_\eta^2 &\le \varepsilon\|f*R\|_\eta^2+C_\varepsilon\|f\|_\eta^2+\varepsilon\sum_{|J|=k_1}\|f*R_J\|_\eta^2+\dots+\varepsilon\sum_{|J|=k_j}\|f*R_J\|_\eta^2\\ &\quad+\varepsilon\sum_{|J|=k_{j+1}}\|f*R_J\|_\eta^2+\varepsilon\sum_{0<|J|<k_{j+1}}\|f*R_J\|_\eta^2\\ &\quad+C\sum_{0<|J|\le r-|I|}\|f*R_{I+J}\|_\eta^2\\ &\le \varepsilon\|f*R\|_\eta^2+C\sum_{|J|=k_1}\|f*R_J\|_\eta^2+\dots+C\sum_{|J|=k_j}\|f*R_J\|_\eta^2\\ &\quad+\varepsilon\sum_{|J|=k_{j+1}}\|f*R_J\|_\eta^2+\varepsilon\sum_{0<|J|<k_{j+1}}\|f*R_J\|_\eta^2+C_\varepsilon\|f\|_\eta^2\,.\end{aligned}$$

Applying the inductive assumption for multi-indices J with $|J|=k_1$, we get

$$\begin{aligned}\|f*R_I\|_\eta^2 &\le \varepsilon\|f*R\|_\eta^2+C\Big(\varepsilon_1\|f*R\|_\eta^2+\varepsilon_1\sum_{0<|J|<k_1}\|f*R_J\|_\eta^2+C_{\varepsilon_1}\|f\|_\eta^2\Big)\\ &\quad+C\sum_{|J|=k_2}\|f*R_J\|_\eta^2+\dots+C\sum_{|J|=k_j}\|f*R_J\|_\eta^2\end{aligned}$$

$$+\,\varepsilon\sum_{|J|=k_{j+1}}\|f*R_J\|_\eta^2+\varepsilon\sum_{0<|J|<k_{j+1}}\|f*R_J\|_\eta^2+C_\varepsilon\|f\|_\eta^2\,.$$

If we fix ε and next take ε_1 sufficiently small, we obtain

$$\begin{aligned}\|f*R_I\|_\eta^2&\\ &\le 2\varepsilon\|f*R\|_\eta^2+C_2\sum_{|J|=k_2}\|f*R_J\|_\eta^2+\ldots+C_2\sum_{|J|=k_j}\|f*R_J\|_\eta^2\\ &\quad+\varepsilon\sum_{|J|=k_{j+1}}\|f*R_J\|_\eta^2+\varepsilon\sum_{0<|J|<k_{j+1}}\|f*R_J\|_\eta^2+C_\varepsilon\|f\|_\eta^2\,.\end{aligned}$$

Proceeding analogously for J with $|J|=k_2,\ldots,k_j$, we find that for every $\varepsilon>0$ there is a constant C_ε such that

$$\|f*R_I\|_\eta^2\le\varepsilon\|f*R\|_\eta^2+\varepsilon\sum_{|J|=k_{j+1}}\|f*R_J\|_\eta^2+\varepsilon\sum_{0<|J|<k_{j+1}}\|f*R_J\|_\eta^2+C_\varepsilon\|f\|_\eta^2\,.$$

Summing the above inequalities over all I with $|I|=k_{j+1}$, we conclude that for every $\varepsilon>0$ there is a constant C_ε such that for every I with $|I|=k_{j+1}$,

$$\|f*R_I\|_\eta^2\le\varepsilon\|f*R\|_\eta^2+\varepsilon\sum_{0<|J|<k_{j+1}}\|f*R_J\|_\eta^2+C_\varepsilon\|f\|_\eta^2\qquad\text{for } f\in C_c^\infty(G)\,.$$

Note that (4.4) is now a consequence of Lemma (4.10).

5. Semigroups on weighted Hilbert spaces. As in the previous section, for a fixed positive m we write $\eta=\phi^m$ (cf. (2.6)). Let R be a truncated kernel of order r which satisfies (3.17). For $l>0$ define a Hilbert space $\mathcal{V}_{\eta,l}$ as the completion of $C_c^\infty(G)$ in the norm $\|\cdot\|_{\mathcal{V}_{\eta,l}}$, where

$$\|f\|_{\mathcal{V}_{\eta,l}}^2=l\|f\|_\eta^2+\sum_{0\le|I|<r}\|f*R_I\|_\eta^2\,. \tag{5.1}$$

The following proposition has a standard proof.

PROPOSITION (5.2). *$f\in\mathcal{V}_{\eta,l}$ if and only if $f\in\mathcal{H}_\eta$ and $f*R_I\in\mathcal{H}_\eta$ for every I with $0\le|I|<r$, where $f*R_I$ is understood in the sense of distributions.*

LEMMA (5.3). *If $u\in C_c^\infty(G)$ then*

$$\begin{aligned}(u\eta)*R(x)&\\ &=\eta(x)(f*R)(x)+\sum_{0<|I|<r}\frac{1}{I!}X^I\eta(x)(f*R_I)(x)+\eta(x)Hf(x)\,,\end{aligned}\tag{5.4}$$

where H is a compactly supported bounded linear operator on $L^2(G)$.

Proof. Using the Taylor expansion of η at x (cf. (2.1)) we get (5.4), where $Hf(x) = (\eta(x))^{-1}\langle \check{R}, f(x\,\cdot\,)\eta^{(x)}(\cdot)\rangle$. Of course by (2.3) and (2.8), H is compactly supported and bounded on $L^2(G)$.

Let us define a sesquilinear form a on $\mathcal{V}_{\eta,l}$ by

$$a(u,v) = \int_G u * R(x)\overline{((v\eta) * R(x))}\,dx + (u*\omega, v)_\eta \qquad \text{for } u,v \in C_c^\infty(G)\,, \tag{5.5}$$

where ω is the function defined in (3.15).

It is now clear from (5.3), (5.2) and (4.6) that for every l there is a constant C_l such that

$$|a(u,v)| \le C_l \|u\|_{\mathcal{V}_{\eta,l}} \|v\|_{\mathcal{V}_{\eta,l}}\,.$$

Let A^η be the operator defined by the form a (cf. Section 2 for the definition) with $\mathcal{V} = \mathcal{V}_{\eta,l}$, $\mathcal{H} = \mathcal{H}_\eta$. Note that A^η does not depend on l.

In order to prove that A^η is a generator of a holomorphic semigroup of operators on $\mathcal{H}_\eta$ in the sector $\Delta_{\pi/2}$, define for $\lambda > 0$ a new form a_λ by

$$a_\lambda(u,v) = a(u,v) + \lambda(u,v)_\eta\,.$$

The operator A_λ^η corresponding to a_λ is $A^\eta + \lambda\,\mathrm{Id}$. By Lemma (5.3), Theorem (4.3) and Lemma (2.8), for every ε there are $l, \lambda > 0$ such that

$$\operatorname{Re} a_\lambda(u,u) \ge \frac{1}{2}\|u\|^2_{\mathcal{V}_{\eta,l}}, \qquad |\operatorname{Im} a_\lambda(u,u)| \le \varepsilon \|u\|^2_{\mathcal{V}_{\eta,l}}\,. \tag{5.7}$$

Theorem (2.12) and (5.7) lead to

THEOREM (5.8). *For every η with $\eta = \phi^m$, the operator A^η is the generator of a holomorphic semigroup of operators on $\mathcal{H}_\eta$ in the sector $\Delta_{\pi/2}$.*

PROPOSITION (5.9). *$f \in \mathcal{D}(A^\eta)$ if and only if $f \in \mathcal{V}_{\eta,l}$ and $f * R^2 \in \mathcal{H}_\eta$, where $f * R^2$ is understood in the sense of distributions.*

COROLLARY (5.10). *If $m_1 \ge m_2 \ge 0$ and $\eta_1 = \phi^{m_1}$, $\eta_2 = \phi^{m_2}$, then $\mathcal{D}(A^{\eta_1}) \subset \mathcal{D}(A^{\eta_2})$, $A^{\eta_1} f = A^{\eta_2} f$ for $f \in \mathcal{D}(A^{\eta_1})$, $T_z^{\eta_1} f = T_z^{\eta_2} f$ for $f \in \mathcal{H}_{\eta_1}$, where $T_z^{\eta_j}$ is the holomorphic semigroup generated by A^{η_j}, $j = 1, 2$.*

PROPOSITION (5.11). *For every weight $\eta = \phi^m$ and every positive integer N the operator $(A^\eta)^N$ is the closure of L^N considered in $C_c^\infty(G)$ in $\mathcal{H}_\eta$ topology.*

Proof. Since L^N is a Rockland operator we can associate with L^N a family of semigroups defined by appropriate forms (cf. (5.5)). So the proof of Proposition (5.11) will be complete if we show that our assertion holds for $N = 1$. For $m_1 > m$ put $\eta_1 = \phi^{m_1}$. Let $\lambda > 0$ be such that $\lambda\,\mathrm{Id} + A^\eta$ and $\lambda\,\mathrm{Id} + A^{\eta_1}$ are invertible in $\mathcal{H}_\eta$ and $\mathcal{H}_{\eta_1}$ respectively. It suffices to prove that

$$(\lambda\,\mathrm{Id} + L)(C_c^\infty(G)) \text{ is dense in } \mathcal{H}_\eta\,. \tag{5.12}$$

Define

$$S^\infty_{\eta_1} = \{f \in \mathcal{H}_{\eta_1} : X^I f \in \mathcal{H}_{\eta_1}\} \subset S^\infty_\eta = \{f \in \mathcal{H}_\eta : X^I f \in \mathcal{H}_\eta\}.$$

First we show that

$$(5.13) \qquad (\lambda \operatorname{Id} + L)(S^\infty_\eta) \text{ is dense in } \mathcal{H}_\eta .$$

Let $f \in C^\infty_{\mathrm{c}}(G)$. By Corollary (5.10), $g = (\lambda \operatorname{Id} + A^{\eta_1})^{-1} f = (\lambda \operatorname{Id} + A^{\eta})^{-1} f \in \mathcal{H}_{\eta_1} \subset \mathcal{H}_\eta$. Moreover, $Y^I g = (\lambda \operatorname{Id} + A^{\eta_1})^{-1} Y^I f \in \mathcal{H}_{\eta_1}$. Since $X^I = \sum_{\|J\| \le \|I\|} w_J Y^J$, where w_J are polynomials (cf. [FS, p. 26]), we see that $X^I g \in \mathcal{H}_\eta$, and consequently $g \in S^\infty_\eta$. Hence (5.13) is proved.

For $f \in S^\infty_\eta$ put $f_n(x) = f(x)\gamma_n(x) \in C^\infty_{\mathrm{c}}(G)$, where $\gamma_n(x) = \gamma(\delta_{n^{-1}} x)$, $\gamma \in C^\infty_{\mathrm{c}}(G)$, $\gamma \equiv 1$ in a neighborhood of 0. Clearly, by the Leibniz formula and the Lebesgue Convergence Theorem

$$\lim_{n\to\infty} (\|f - f_n\|_\eta + \|Lf - Lf_n\|_\eta) = 0\,,$$

which ends the proof of (5.12).

Corollary (5.14). *For every z with $\operatorname{Re} z > 0$, and every left-invariant differential operator ∂ there is a constant $C_{\eta,z}$ such that*

$$(5.15) \qquad \|(\partial T^\eta_z f)\sqrt{\eta}\|_{L^\infty} \le C_{\eta,z} \|f\|_\eta \quad \textit{for } f \in \mathcal{H}_\eta .$$

Proof. Let $f \in \mathcal{H}_\eta$. Since T^η_z is holomorphic, we obtain $T^\eta_z f \in \bigcap_n \mathcal{D}((A^\eta)^n)$ and $\|(A^\eta)^n T^\eta_z f\|_\eta \le C\|f\|_\eta$. Using (2.8), (5.11), (1.1), and Sobolev estimates, we get (5.15).

Proof of Theorem (1.3). For the fact that the semigroup is holomorphic on weighted Hilbert spaces in the sector $\Delta_{\pi/2}$ see Theorem (5.8) and Lemma (2.7).

By the spectral theorem, Proposition (5.11) and estimates (1.1), for every left-invariant differential operator ∂ and every z with $\operatorname{Re} z > 0$, there are constants M, C such that

$$(5.16) \qquad \begin{aligned} \|\partial T_z f\|_{L^2} &\le C\|(\operatorname{Id} + \overline{L})^M T_2 f\|_{L^2} \\ &\le C\Big\| \int_0^\infty (1+\lambda)^M e^{-z\lambda}\, dE_{\overline{L}}(\lambda) f \Big\|_{L^2} \le C\|f\|_{L^2}\,, \end{aligned}$$

where $E_{\overline{L}}$ is the spectral resolution for $\overline{L}$.

Using Sobolev estimates, we have

$$(5.17) \qquad |T_z f(0)| \le C\|f\|_{L^2} .$$

Since T_z commutes with left translations, we deduce from (5.17) that there is a function $p_z \in L^2(G)$ such that

$$T_z f = f * p_z .$$

Note that for $t > 0$, p_t is real and symmetric. By virtue of Corollary (5.14) the proof of our theorem will be completed if we show that $p_t \in \mathcal{H}_\eta$ for every $\eta = \phi^m$.

Let Γ and ψ be as in Lemma (2.9). Fix $\eta = \phi^m$, $\eta_1 = \phi^{2m+2}$. Note that $p_t = p_{t/2} * p_{t/2} \in L^\infty(G)$. Hence there is a constant C_0 such that for every $b \in \Gamma$

$$\|\psi^2 \lambda_{b^{-1}} p_t\|_{\eta_1} \le C_0 \,. \tag{5.18}$$

By (5.15) we get

$$\begin{aligned} \Big| \int (\psi_b^2 p_t)(x) p_t(x)\, dx \Big| \eta(b) &= |T_t^{\eta_1}(\psi_b^2 p_t)(0)|\, \eta(b) \\ &= |T_t^{\eta_1}(\psi^2 \lambda_{b^{-1}} p_t)(b)|\, \eta(b) \\ &\le C_{\eta_1, t} C_0 \phi^{-1}(b) \,. \end{aligned} \tag{5.19}$$

Now by Lemma (2.8), Corollary (2.11), and (5.19) we obtain

$$\begin{aligned} \int |p_t(x)|^2 \eta(x)\, dx &\le \sum_{b\in\Gamma} \int (p_t \psi_b^2)(x) p_t(x) \eta(x)\, dx \\ &\le CC_0 \sum_{b\in\Gamma} \Big(\int (p_t \cdot \psi_b^2)(x) p_t(x)\, dx \Big) \eta(b) \\ &\le C \sum_{b\in\Gamma} \phi^{-1}(b) < \infty \,, \end{aligned}$$

which completes the proof.

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