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ON COMPLETE ORBIT SPACES OF SL(2) *ACTIONS, II*

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The aim of this paper is to extend the results of [BB-Ś2] concerning geometric quotients of actions of SL(2) to the case of good quotients. Thus the results of the present paper can be applied to any action of SL(2) on a complete smooth algebraic variety, while the theorems proved in [BB-Ś2] concerned only special situations.

Like in [BB-Ś2], the source of our study lies in Mumford's Geometric Invariant Theory [GIT]. His results concerning semi-stability lead to the Conjecture (see below). In order to state it we need the following definition:

DEFINITION. Let T be an algebraic torus and let U, V be two open T-invariant subsets of X for which there exist good quotients $\pi_U : U \to U//T$ and $\pi_V : V \to V//T$. We shall write $V \lhd U$ if $V \subset U$ and the induced morphism $V//T \to U//T$ is an open embedding.

We shall say that a T-invariant open subset U of X having a good quotient is *maximal* with respect to the property of having good quotient if U is maximal with respect to $\lhd$.

CONJECTURE. Let X be a smooth algebraic variety with an action of a reductive group G. Let T be a maximal torus of G and let $N(T)$ be its normalizer in G. Let U be an $N(T)$-invariant open subset of X for which there exists a good quotient $\pi : U \to U//T$ and which is maximal with respect to this property. Then $\bigcap_{g\in G} gU$ is open, G-invariant and there exists a good quotient $\bigcap_{g\in G} gU \to \bigcap_{g\in G} gU//G$. Moreover, if $U//T$ is complete, then $\bigcap_{g\in G} gU//G$ is also complete.

In the present paper we only consider the case $G = \mathrm{SL}(2)$. Theorem 1 shows that if $U//T$ is projective then the conjecture is valid. Moreover, then X and $\bigcap_{g\in \mathrm{SL}(2)} gU//G$ are projective and there exists an ample, invertible, G-linearized sheaf $\mathcal{L}$ on X such that U is the set of semi-stable points with respect to the action of T induced by the action of G.

We also prove the conjecture under the additional assumption that either $U//T$ is complete (Theorem 2) or $U//T$ is quasi-projective (Theorem 9).

Answering a question of D. Luna we also describe an example of an action of SL(2) on an algebraic variety X such that there exists a geometric quotient $X \to X/\mathrm{SL}(2)$, where $X/\mathrm{SL}(2)$ is an algebraic space but not an algebraic variety.

1. Notation and terminology. We use the terminology of [BB-Ś1] and [BB-Ś2]. We now fix the notation and quote the definitions needed in the sequel.

The ground field k is supposed to be algebraically closed of characteristic 0.

If $X \to Y$ is a good quotient of X by an action of a reductive group G, then we write $X/\!/G$ in place of Y. We write X/G for the geometric quotient space of X by the action of G.

For a given action of a one-dimensional torus $T = k^*$ on a smooth complete variety X we denote by X^T the fixed point subvariety of the action. Let $X^T = X_1 \cup \ldots \cup X_r$ be the decomposition into irreducible components. For $i = 1, \ldots, r$, we define

$$X_i^+ = \{x \in X; \lim_{t\to 0} tx \in X_i\}, \qquad X_i^- = \{x \in X; \lim_{t\to\infty} tx \in X_i\}.$$

We say that X_i is *less* than X_j, and write $X_i \prec X_j$, if there exists a finite sequence of points $x_1, \ldots, x_m \in X - X^T$ such that

(a) $\lim_{t\to 0} tx_1 \in X_i$,

(b) $\lim_{t\to\infty} tx_m \in X_j$,

(c) for $k = 1, \ldots, m-1$, $\lim_{t\to\infty} tx_k$ and $\lim_{t\to 0} tx_{k+1}$ belong to the same irreducible component of X^T.

By a *semi-section* of $\{X_1, \ldots, X_r\}$ we mean a partition, denoted by A, of $\{X_1, \ldots, X_r\}$ into three pairwise disjoint subsets A^-, A^0, A^+ such that $A^- \neq \emptyset \neq A^+$, and if $X_i \in A^- \cup A^0$, $X_j \prec X_i$ and $i \neq j$, then $X_j \in A^-$. Any semi-section determines two open T-invariant subsets

$$X^{\mathrm{ss}}(A) = X - \Big(\bigcup_{j\in A^-} X_j^- \cup \bigcup_{j\in A^+} X_j^+\Big), \qquad X^{\mathrm{s}}(A) = X^{\mathrm{ss}} - \bigcup_{j\in A^0} (X_j^- \cup X_j^+),$$

where we write $j \in A^-, A^0, A^+$ in place of $X_j \in A^-, A^0, A^+$. We shall call $X^{\mathrm{ss}}(A)$ and $X^{\mathrm{s}}(A)$ the sets of *semi-stable* and *stable* points determined by the semi-section A, respectively.

It has been proved in [BB-Ś1] that for any semi-section A there exists a good quotient $\pi : X^{\mathrm{ss}}(A) \to X^{\mathrm{ss}}(A)/\!/T$, where $X^{\mathrm{ss}}(A)/\!/T$ is a complete algebraic variety, $\pi \mid X^{\mathrm{s}}(A)$ is a geometric quotient and $\pi(X^{\mathrm{s}}(A))$ is an open subset of $X^{\mathrm{ss}}(A)/\!/T$.

Let X be a smooth complete algebraic variety with a non-trivial action of SL(2). Assume that

$$T=\left\{\begin{bmatrix} t & 0\\ 0 & t^{-1}\end{bmatrix}; t\in k^*\right\},$$

$$B^+=\left\{\begin{bmatrix} t & \lambda\\ 0 & t^{-1}\end{bmatrix}; t\in k^*, \lambda\in k\right\},\qquad B^-=\left\{\begin{bmatrix} t & 0\\ \lambda & t^{-1}\end{bmatrix}; t\in k^*, \lambda\in k\right\},$$

$$N(T)=T\cup\tau T,\quad \text{where}\quad \tau=\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}.$$

Then T is a maximal torus, $N(T)$ is the normalizer of T and B^+, B^- are two Borel subgroups containing T.

The Weyl group $W=N(T)/T$ acts on $\{X_1,\ldots,X_r\}$. Denote by w the involution on $\{X_1,\ldots,X_r\}$ determined by τ. A semi-section (A^-, A^0, A^+) for the action of T is called *Weyl-invariant* if $w(A^-)=A^+$ (hence $w(A^0)=A^0$, $w(A^+)=A^-$).

2. Projective and complete quotients. The proof of Theorem 1 in [BB-Ś1] can be easily adapted to give the proof of the following Theorem 1.

THEOREM 1. *Let $U\subset X$ be an $N(T)$-invariant open subset such that a good quotient $U\to U//T$ exists and $U//T$ is projective. Then X is projective and there exists an ample* SL(2)*-linearized linear sheaf $\mathcal{L}$ on X such that*

$$X^{\mathrm{ss}}(\mathcal{L})=\bigcap_{g\in\mathrm{SL}(2)} gU\,.$$

Hence $\bigcap_{g\in\mathrm{SL}(2)} gU$ is open and SL(2)*-invariant, a good quotient*

$$\bigcap_{g\in\mathrm{SL}(2)} gU\to\bigcap_{g\in\mathrm{SL}(2)} gU//\mathrm{SL}(2)$$

exists and $\bigcap_{g\in\mathrm{SL}(2)} gU//\mathrm{SL}(2)$ is a projective (*normal*) *variety.*

THEOREM 2. *Let $U\subset X$ be an $N(T)$-invariant open subset of X such that a good quotient $U\to U//T$ exists and $U//T$ is a complete algebraic variety. Then a good quotient*

$$\bigcap_{g\in\mathrm{SL}(2)} gU\to\bigcap_{g\in\mathrm{SL}(2)} gU//\mathrm{SL}(2)$$

exists and $\bigcap_{g\in\mathrm{SL}(2)} gU//\mathrm{SL}(2)$ is a complete normal algebraic space.

In the proof we may and will assume that $U=X^{\mathrm{ss}}(A)$, where A is a Weyl-invariant semi-section (see [BB-Ś1]).

First we prove the following:

PROPOSITION 3. *Let $A=(A^-, A^0, A^+)$ be a Weyl-invariant semi-section and let $X^{\mathrm{ss}}(A), X^{\mathrm{s}}(A)$ be the sets of semi-stable and stable points determined* *by A. Then*

$$\mathrm{SL}(2)(X^{\mathrm{ss}}(A) - X^{\mathrm{s}}(A)) \subset X^{\mathrm{ss}}(A)\,.$$

Proof. We shall write $X^{\mathrm{ss}}, X^{\mathrm{s}}$ instead of $X^{\mathrm{ss}}(A)$ and $X^{\mathrm{s}}(A)$, respectively. Let $x \in X^{\mathrm{ss}} - X^{\mathrm{s}}$. Then $x \in X_i^+ \cup X_i^-$ for some $X_i \in A^0$. By symmetry we may assume that $x \in X_i^+$. Assume that $gx \notin X^{\mathrm{ss}}$ for some $g \in \mathrm{SL}(2)$. Then $gx \in \bigcup X_l^- \cup \bigcup X_l^+$. We may suppose that $gx \in X_l^+$ for some $X_l \in A^+$ (otherwise we take τg instead of g). By the Bruhat decomposition, $g = b_1 \tau b_2$ for some $b_1, b_2 \in B^+$. Now $b_2 x \in X_i^+$ and $\tau b_2 x \in X_l^+$ (since $B^+ x \subset X_i^+$ and $B^+ X_l^+ \subset X_l^+$, see [C-S]). Consider $\lim_{t\to\infty} t\tau b_2 x$. We get

$$\lim_{t\to\infty} t\tau b_2 x = \lim_{t\to\infty} \tau(\tau^{-1} t \tau) b_2 x = \tau \lim_{t\to\infty} t^{-1} b_2 x = \tau \lim_{t\to 0} t b_2 x\,,$$

hence $\tau b_2 x \in (\tau X_i)^-$, where $\tau X_i \in A^0$. At the same time $\tau b_2 x \in X_l^+$, which implies that $X_l \prec \tau X_i \in A^0$. This contradicts the assumption $X_l \in A^+$. The proof of the proposition is complete.

Now we shall prove the first part of Theorem 2. Let $V = \bigcap_{g\in \mathrm{SL}(2)} gX^{\mathrm{ss}}$. Then V is obviously SL(2)-invariant. In order to see that V is open notice that

$$\begin{aligned} X - V = \mathrm{SL}(2)(X - X^{\mathrm{ss}}) &= \mathrm{SL}(2)\Big(\bigcup_{l\in A^-} X_l^- \cup \bigcup_{l\in A^+} X_l^+\Big) \\ &= \mathrm{SL}(2)\Big(\bigcup_{l\in A^-} X_l^-\Big) \cup \mathrm{SL}(2)\Big(\bigcup_{l\in A^+} X_l^+\Big)\,. \end{aligned}$$

Hence it suffices to show that $\mathrm{SL}(2)(\bigcup_{l\in A^-} X_l^-)$ and $\mathrm{SL}(2)(\bigcup_{l\in A^+} X_l^+)$ are closed. This is clear, since $\mathrm{SL}(2)/B^-$, $\mathrm{SL}(2)/B^+$ are complete and $\bigcup_{l\in A^-} X_l^-$ and $\bigcup_{l\in A^+} X_l^+$ are closed and invariant under the actions of B^- and B^+, respectively.

Now in order to show that there exists a good quotient $\pi : V \to V/\!/\mathrm{SL}(2)$ it suffices to prove that there exists a good quotient $\pi_T : V \to V/\!/T$ (by Theorem 1 of [BB-Ś4] or Theorem 5 of [BB-Ś3]). This is obvious since V is an open, T-invariant and T-saturated subset of X^{ss}. In fact, every T-orbit contained in V is either closed in X^{ss} or belongs to $X^{\mathrm{ss}} - X^{\mathrm{s}}$. In the second case, by Proposition 3 the closure in X^{ss} of the orbit is contained in V.

It remains to show that $V/\!/\mathrm{SL}(2)$ is complete. We start with the following remark:

Let U_1, U_2 be two different Weyl-invariant semi-sectional sets defined by semi-sections (A_1^-, A_1^0, A_1^+), (A_2^-, A_2^0, A_2^+), respectively. We say that U_2 is an *elementary transform* of U_1 if there exists a maximal (with respect to the order $\prec$ given by the action of T) element X_{i_0} in A_2^- such that

$$A_1^0 = A_2^0 \cup \{X_{i_0}, wX_{i_0}\} \quad \text{and} \quad A_1^- = A_2^- - \{X_{i_0}\}\,.$$

Notice that in this case $(X_{i_0})^{B^+} = \emptyset$. If U_2 is an elementary transform of U_1 then there is a morphism α of $Y_2 = \bigcap_{g\in \mathrm{SL}(2)} gU_2//\mathrm{SL}(2)$ into $Y_1 = \bigcap_{g\in \mathrm{SL}(2)} gU_1//\mathrm{SL}(2)$. It follows from [BB-Ś2] that $\mathrm{SL}(2)(X_{i_0}^+ - B^+X_{i_0})$ is a closed subset of $\bigcap_{g\in \mathrm{SL}(2)} gU_2$. Similarly, by Proposition 3, $\mathrm{SL}(2)(X_{i_0}^+ \cup X_{i_0}^-)$ is a closed subset of $\bigcap_{g\in \mathrm{SL}(2)} gU_1$, since $X_{i_0}^+, X_{i_0}^-$ are B^+- and B^--invariant, respectively, and $\mathrm{SL}(2)/B^+, \mathrm{SL}(2)/B^-$ are complete. The morphism α restricted to $Y_2 - (\mathrm{SL}(2)(X_{i_0}^+ - B^+X_{i_0})//\mathrm{SL}(2))$ is an isomorphism onto $Y_1 - (\mathrm{SL}(2)(X_{i_0}^+ \cup X_{i_0}^-)//\mathrm{SL}(2))$.

Moreover, notice that

$$Z_1 = \mathrm{SL}(2)(X_{i_0}^+ \cup X_{i_0}^-)/\mathrm{SL}(2) \quad \text{and} \quad Z_2 = \mathrm{SL}(2)(X_{i_0}^+ - B^+X_{i_0})/\mathrm{SL}(2)$$

are complete. In fact, Z_2 is complete by Lemma 1 and Corollary 1 in [BB-Ś2]. Moreover, X_{i_0} is contained in $\mathrm{SL}(2)(X_{i_0}^+ \cup X_{i_0}^-)$, the quotient morphism $\mathrm{SL}(2)(X_{i_0}^+ \cup X_{i_0}^-) \to Z_2$ maps X_{i_0} onto Z_2 and X_{i_0} is complete, hence Z_1 is complete.

LEMMA 4. *If U_2 is an elementary transform of U_1, then Y_2 is complete iff Y_1 is complete.*

Proof. If Y_2 is complete, then its image $\alpha(Y_2)$ in Y_1 is complete. Since α is an isomorphism of open sets, it follows that α is onto, and Y_1 is complete.

Conversely, assume that Y_1 is complete. We noticed earlier that $\alpha \mid Y_2 - Z_2$ is an isomorphism onto $Y_1 - Z_1$ and Z_1, Z_2 are complete. Moreover, Z_2 is connected. From Lemma 3 in [BB-Ś2] it follows that Y_2 is complete.

LEMMA 5. *Let U_1 and U_2 be two semi-sectional $N(T)$-invariant sets in X. Then there exists a chain of semi-sectional $N(T)$-invariant sets $V_1 = U_1$, $V_2, \ldots, V_k = U_2$ such that for each $i = 1, \ldots, k-1$, either V_i is an elementary transform of V_{i+1}, or vice versa.*

Proof. We use the method of the proof of Lemma 2 in [BB-Ś2].

LEMMA 6. *Let $\beta : X_1 \to X$ be an* SL(2)*-equivariant birational morphism of smooth algebraic complete varieties and let $U \subset X$ be an $N(T)$-invariant semi-sectional set. Moreover, assume that X_1 is projective. Then there exists an $N(T)$-invariant semi-sectional set $W \subset \beta^{-1}(U)$.*

Proof. Assume that U is a semi-sectional set corresponding to a Weyl-invariant semi-section (A^-, A^0, A^+). We shall define a semi-section (A_1^-, A_1^0, A_1^+) in the set of connected components of $(X_1)^T$. We may decompose A^0 into disjoint subsets $S_1 \cup S_2 \cup wS_2$ in such a way that

$$X_i \in S_1 \quad \text{iff} \quad wX_i = X_i$$

(of course this decomposition is not uniquely defined). For any $X_i \in S_1$ let (D_i^-, D_i^0, D_i^+) be any Weyl-invariant semi-section in the set of connected components of $\beta^{-1}(X_i)$. Such a semi-section exists because $\beta^{-1}(X_i)$ is projective. Let $X_{1,j}$ be a connected component of $(X_1)^T$. Then define A_1^+ in the following way: $X_{1,j} \in A_1^+$ iff any of the following conditions is satisfied:

(a) $\beta(X_{1,j}) \subset X_i$ where $X_i \in A^+$,
(b) $\beta(X_{1,j}) \subset X_j$ where $X_j \in S_2$,
(c) $X_{1,j} \in D_i^+$.

The set A_1^0 is defined by

$$X_{1,j} \in A_1^0 \ \Leftrightarrow \ X_{1,j} \in D_i^0 \,.$$

It is easy to check that the partition (A_1^-, A_1^0, A_1^+) (where $A_1^- = wA_1^+$) of the set of connected components of $(X_1)^T$ is a Weyl-invariant semi-section. Obviously the semi-sectional set defined by this semi-section is contained in $\beta^{-1}(U)$.

We are now ready to prove the second part of Theorem 2. Assume first that X is projective. Then there exists an $N(T)$-invariant semi-sectional set $U_1 = X_T^{\mathrm{ss}}(\mathcal{L})$ of semi-stable points with respect to some T-linearized ample sheaf $\mathcal{L}$, where the T-linearization is induced by an SL(2)-linearization. Then $\bigcap_{g\in\mathrm{SL}(2)} gU_1/\!/\mathrm{SL}(2)$ is complete. By Lemmas 4 and 5, for any $N(T)$-invariant semi-sectional set U in X, the quotient $\bigcap_{g\in\mathrm{SL}(2)} gU/\!/\mathrm{SL}(2)$ is complete.

If X is complete and not projective, then by the equivariant Chow Lemma (see [S]) there exists a projective variety X_1 and an SL(2)-equivariant birational morphism $\beta : X_1 \to X$. Let U be an $N(T)$-invariant semi-sectional set in X and let $U_1 \subset X_1$ be an $N(T)$-invariant semi-sectional set contained in $\beta^{-1}(U)$. Such a set exists by Lemma 6. Since Y is projective, $\bigcap_{g\in\mathrm{SL}(2)} gU_1/\!/\mathrm{SL}(2)$ is complete. The morphism $\beta \mid \bigcap_{g\in\mathrm{SL}(2)} gU_1 : \bigcap_{g\in\mathrm{SL}(2)} gU_1 \to \bigcap_{g\in\mathrm{SL}(2)} gU$ is birational and SL(2)-equivariant, hence it induces a birational morphism $\bigcap_{g\in\mathrm{SL}(2)} gU_1/\!/\mathrm{SL}(2) \to \bigcap_{g\in\mathrm{SL}(2)} gU_2/\!/\mathrm{SL}(2)$. Since $\bigcap_{g\in\mathrm{SL}(2)} gU_1/\!/\mathrm{SL}(2)$ is complete, it follows that $\bigcap_{g\in\mathrm{SL}(2)} gU_2/\!/\mathrm{SL}(2)$ is also complete. This completes the proof of Theorem 2.

THEOREM 7. *Let $A_1 = (A_1^-, A_1^0, A_1^+)$, $A_2 = (A_2^-, A_2^0, A_2^+)$ be two different Weyl-invariant semi-sections. Then $\bigcap_{g\in\mathrm{SL}(2)} gX^{\mathrm{ss}}(A_1) \neq \bigcap_{g\in\mathrm{SL}(2)} gX^{\mathrm{ss}}(A_2)$ unless both intersections are empty.*

Proof. If $A_1^0 \neq A_2^0$, then the theorem follows from Proposition 3. Assume that $A_1^0 = A_2^0$ and let $X_{i_0} \in A_1^- - A_2^-$ be maximal in A_1^- with respect to the order $\prec$ induced by T. Then $X_{i_0} \in A_2^+$. In this case we use the same argument as in the proof of Lemma 5 in [BB-Ś2] to see that $\mathrm{SL}(2)x \subset X^{\mathrm{ss}}(A_1)$ for any $x \in X_{i_0}^+ - B^+X_{i_0}$. But obviously $x \notin X^{\mathrm{ss}}(A_2)$. Similarly, if $x \in X_{i_0} - B^-X_{i_0}$, then $\mathrm{SL}(2)x \subset X^{\mathrm{ss}}(A_2)$, but $x \notin X^{\mathrm{ss}}(A_1)$. So it remains to consider the case where $X_{i_0}^+ = B^+X_{i_0}$ and $X_i^- = B^-X_{i_0}$.

But then $\mathrm{SL}(2)X_{i_0}$ is dense in X. On the other hand, the considered intersections are open and disjoint from $\mathrm{SL}(2)X_{i_0}$. Hence they are empty. This completes the proof of the theorem.

THEOREM 8. *Let* $\mathrm{Pic}(x) = \mathbb{Z}$ *and let* X *be projective. Then there exists the greatest open* $\mathrm{SL}(2)$*-invariant subset* V *of* X *such that there exists a good quotient* $V \to V/\!/\mathrm{SL}(2)$, *where* $V/\!/\mathrm{SL}(2)$ *is an algebraic variety. Moreover,* $V/\!/\mathrm{SL}(2)$ *is projective.*

Proof. Let V be the $\mathrm{SL}(2)$-invariant open set of points satisfying the following condition: $x \in V$ if and only if there exists an affine open $\mathrm{SL}(2)$-invariant neighbourhood U of x. We shall show that $V = X^{\mathrm{ss}}(\mathcal{L})$, for some ample $\mathrm{SL}(2)$-linearized sheaf $\mathcal{L}$. Notice that since $\mathrm{Pic}(X) = \mathbb{Z}$, an invertible sheaf $\mathcal{F}$ is ample iff it has a non-zero section with support different from X.

Fix any invertible ample sheaf $\mathcal{L}$ on X. Let $x \in V$ and let U be any affine $\mathrm{SL}(2)$-invariant neighbourhood of x. Since there exists $U \to U/\!/\mathrm{SL}(2)$ and $U/\!/\mathrm{SL}(2)$ is affine, by [GIT] there exists an invertible $\mathrm{SL}(2)$-linearized sheaf $\mathcal{L}_1$ on U such that $U = U^{\mathrm{ss}}(\mathcal{L}_1)$. Let $s \in H^0(U, \mathcal{L}_1)^{\mathrm{SL}(2)}$ be a section such that $s(x) \neq 0$ and its support is affine. The sheaf $\mathcal{L}_1$ can be extended to an invertible $\mathrm{SL}(2)$-linearized sheaf $\mathcal{L}_2$ on X such that s extends to a section of $\mathcal{L}_2$ on X, equal to 0 on $X - U$ (see the proof of Prop. 1.13 in [GIT]). Then $\mathcal{L}_2$ is ample on X and $U \subset X^{\mathrm{ss}}(\mathcal{L}_2)$. Since $\mathrm{Pic}(X) = \mathbb{Z}$, $\mathcal{L}^{\otimes n} = \mathcal{L}_2^{\otimes m}$ for some positive integers n, m. Thus $X^{\mathrm{ss}}(\mathcal{L}) = X^{\mathrm{ss}}(\mathcal{L}_2)$ and $U \subset X^{\mathrm{ss}}(\mathcal{L})$. The proof is complete.

EXAMPLE. Now we shall construct an example of a smooth projective algebraic variety X with an action of $\mathrm{SL}(2)$ and an open $\mathrm{SL}(2)$-invariant subset U of X such that there exists a geometric quotient $U \to U/\mathrm{SL}(2)$, where $U/\mathrm{SL}(2)$ is a complete algebraic space which is not an algebraic variety. This gives a negative answer to a question of D. Luna.

It is enough to describe a projective smooth algebraic variety X with an action of $\mathrm{SL}(2)$ such that $\mathrm{Pic}(X) = \mathbb{Z}$ and which has two different $N(T)$-invariant sectional sets V_1, V_2 such that $\bigcap_{g \in \mathrm{SL}(2)} gV_1 \neq \emptyset \neq \bigcap_{g \in \mathrm{SL}(2)} gV_2$.

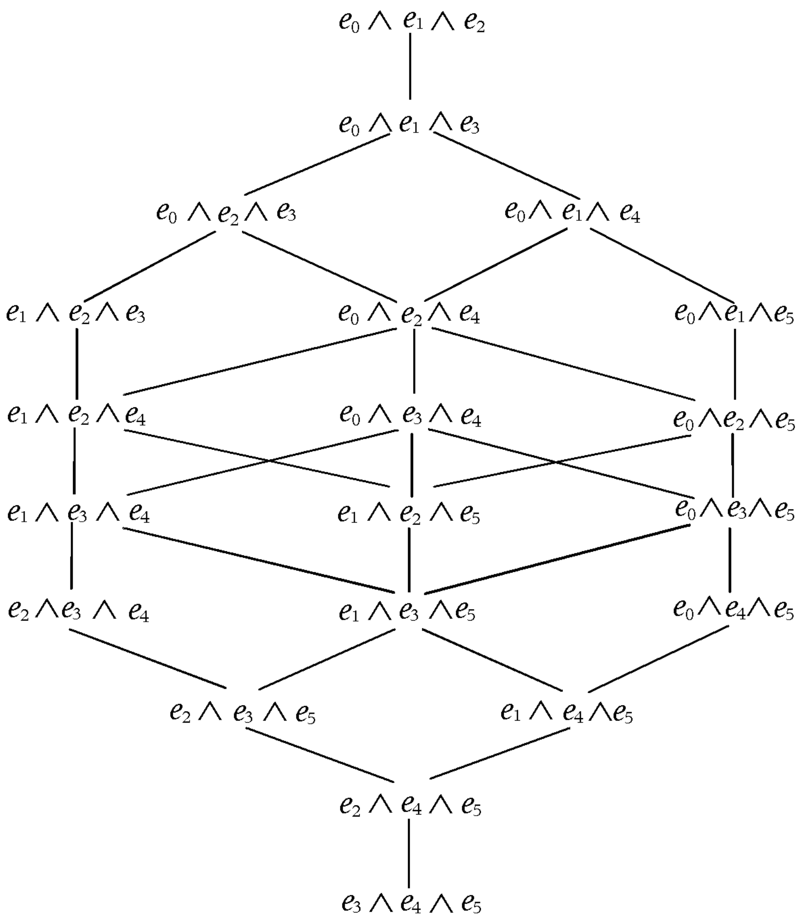
In fact, by Theorem 7, $\bigcap_{g \in \mathrm{SL}(2)} gV_1 \neq \bigcap_{g \in \mathrm{SL}(2)} gV_2$ and by Theorem 8, at most one of the sets $\bigcap_{g \in \mathrm{SL}(2)} gV_i/\mathrm{SL}(2)$, $i = 1, 2$, is an algebraic variety.

Let X be the Grassmannian of 3-dimensional linear subspaces in a 6-dimensional linear space V with an action of $\mathrm{SL}(2)$ induced by an irreducible representation of $\mathrm{SL}(2)$ in V. Then V can be identified with the space of 5-forms in two variables x, y, with the action of $\mathrm{SL}(2)$ induced by the natural representation of $\mathrm{SL}(2)$ in the two-dimensional space of linear forms in x, y. Set

$$e_0 = x^5, \quad e_1 = x^4 y, \quad e_2 = x^3 y^2, \quad e_4 = x y^4, \quad e_5 = y^5 .$$

Then the action of $t \in T$ is given by $t(e_i) = t^{5-2i} e_i$ and $\tau(e_i) = e_{5-i}$, for

$i = 0, 1, \ldots, 5$. The fixed points of the action of T on X are of the form $e_i \wedge e_j \wedge e_k$, $i < j < k$, $i, j, k = 0, 1, \ldots, 5$, with the order described by the diagram.



It is clear that we have two $N(T)$-invariant sectional sets given by the following sections:

$$\text{a)}\quad A_1^+ = \{e_0 \wedge e_1 \wedge e_2,\ e_0 \wedge e_1 \wedge e_3,\ e_0 \wedge e_2 \wedge e_3,\ e_0 \wedge e_1 \wedge e_4,\ e_1 \wedge e_2 \wedge e_3,\ e_0 \wedge e_2 \wedge e_4,\ e_0 \wedge e_1 \wedge e_5,\ e_1 \wedge e_2 \wedge e_4,\ e_0 \wedge e_2 \wedge e_5,\ e_0 \wedge e_3 \wedge e_4\},$$

$$A_1^- = \{X_1, \ldots, X_r\} - A_1^+, \quad V_1 = X^{\mathrm{ss}}(A_1^-, A_1^+),$$

$$\text{b)}\quad A_2^+ = (A_1^+ - \{e_0 \wedge e_3 \wedge e_4\}) \cup \{e_1 \wedge e_2 \wedge e_5\},$$

$$A_2^- = \{X_1, \ldots, X_r\} - A_2^+, \quad V_2 = X^{\mathrm{ss}}(A_2^-, A_2^+).$$

Since $\dim\{e_0 \wedge e_3 \wedge e_4\}^+ = 4 > 2$, the set $\{e_0 \wedge e_3 \wedge e_4\}^+ - B^+\{e_0 \wedge e_3 \wedge e_4\}$

is non-empty, and it follows from Proposition 3 that $\bigcap_{g\in \mathrm{SL}(2)} gV_1 \neq \emptyset$. Similarly $\bigcap_{g\in \mathrm{SL}(2)} gV_2 \neq \emptyset$. Hence by Theorem 7, the two intersections are different.

One may easily check that in case a) one obtains the geometric quotient $\bigcap_{g\in \mathrm{SL}(2)} gV_1 \to \bigcap_{g\in \mathrm{SL}(2)} gV_1/\mathrm{SL}(2)$ with projective orbit space. Since $\mathrm{Pic}(X) = \mathbb{Z}$, in case b) one obtains the geometric quotient $\bigcap_{g\in \mathrm{SL}(2)} gV_2 \to \bigcap_{g\in \mathrm{SL}(2)} gV_2/\mathrm{SL}(2)$ with orbit space which is not an algebraic variety.

THEOREM 9. *Let X be a smooth complete algebraic variety with an action of* SL(2). *Let U be an $N(T)$-invariant open subset of X for which there exists a good quotient $U \to U/\!/T$ and let U be maximal with respect to this property. Moreover, assume that $U/\!/T$ is quasi-projective. Then $\bigcap_{g\in \mathrm{SL}(2)} gU$ is open,* SL(2)*-invariant, and there exists a good quotient $\bigcap_{g\in \mathrm{SL}(2)} gU \to \bigcap_{g\in \mathrm{SL}(2)} gU/\!/\mathrm{SL}(2)$.*

The proof of the theorem will follow from a sequence of lemmas.

LEMMA 10. *Under the assumptions of Theorem 9, the set $X - U$ is the union of two closed subsets F_+, F_- such that F_+ is B^+-invariant and F_- is B^--invariant.*

Proof. It follows from [GIT] that there exists an $N(T)$-linearized invertible ample sheaf $\mathcal{L}$ on U such that U consists of semi-stable points with respect to $\mathcal{L}$. We may extend the sheaf $\mathcal{L}$ to X so that $X^{\mathrm{ss}}(\mathcal{L}) \supset U$, $X^{\mathrm{s}}(\mathcal{L}) \supset U^{\mathrm{s}}(\mathcal{L})$. Moreover, we may assume that there exist sections $s_1, \ldots, s_r \in \Gamma(X, \mathcal{L})$ which separate points and tangent vectors. Such an extension can be found using the method of proof of Theorem 1 of [BB-Ś2].

Some tensor power $\mathcal{L}^{\otimes n}, n > 0$, can be SL(2)-linearized (see [GIT]). Since the character group of $N(T)$ is finite, the restriction of the SL(2)-linearization to $N(T)$ coincides with the $N(T)$-linearization determined previously (see [GIT]). It follows from the above that the rational map $\Phi_{\mathcal{L}} : X \to \mathbb{P}^m$ determined by the SL(2)-linearized sheaf $\mathcal{L}$ is SL(2)-equivariant and gives an embedding of U into $\mathbb{P}^m$. Hence for any $g \in \mathrm{SL}(2)$, $\Phi_{\mathcal{L}}|gU$ is also an embedding. It follows that $\Phi_{\mathcal{L}}|\bigcup_{g\in \mathrm{SL}(2)} gU$ is an embedding. In fact, if $x_1, x_2 \in \bigcup_{g\in \mathrm{SL}(2)} gU$, then $x_1 \in g_1U$, $x_2 \in g_2U$ for some $g_1, g_2 \in \mathrm{SL}(2)$. The set of $g \in \mathrm{SL}(2)$ such that $x_1 \in gU$ is not empty and open, and similarly for the set of $g \in \mathrm{SL}(2)$ such that $x_2 \in gU$. Since SL(2) is irreducible as an algebraic variety, these two sets intersect, i.e. there exists $g \in \mathrm{SL}(2)$ such that $x_1, x_2 \in gU$. Thus if $x_1 \neq x_2$, then $\Phi_{\mathcal{L}}(x_1) \neq \Phi_{\mathcal{L}}(x_2)$. Similarly $\Phi_{\mathcal{L}}$ separates tangent vectors.

Let $X_1 = \overline{\Phi_{\mathcal{L}}(X)}$. Then $\Phi_{\mathcal{L}}(U) \subset X_1^{\mathrm{ss}}$. Now we need the following:

LEMMA 11. *Let* SL(2) *act on $\mathbb{P}^m$. Let $U_0 \subset (\mathbb{P}^m)^{\mathrm{ss}}$ be a locally closed T-invariant subset such that a good quotient $U_0 \to U_0/\!/T$ exists. Then there* *exists a semi-sectional set W in $\overline{U}_0$ such that U_0 is saturated in W. If U_0 is $N(T)$-invariant, then W can also be chosen to be $N(T)$-invariant.*

Proof. Let $Y=\overline{U}_0$. The set U_0 is contained in the set $V=Y\cap(\mathbb{P}^m)^{\mathrm{ss}}$ of semi-stable points of Y with respect to the action of T. The set V is semi-sectional, corresponding to a semi-section $A_1=(A_1^-,A_1^0,A_1^+)$ in the set $Y_1,\dots,Y_r$ of connected components of Y^T. We shall define a semi-section $A=(A^-,A^0,A^+)$ such that U_0 is saturated in $W=X^{\mathrm{ss}}(A)$.

Let $Y_i\in A^0$ and $Y_i\cap U_0=\emptyset$. Then there are three possibilities:

$$\text{(i)}\quad Y_0\cap(Y_i^-\cup Y_0^+)=\emptyset,\qquad \text{(ii)}\quad U_0\cap Y_i^-\neq\emptyset,\qquad \text{(iii)}\quad U_0\cap Y_i^+\neq\emptyset\,.$$

We shall define (A^-,A^0,A^+) in the following way: $Y_i\in A_1^-$ implies that $Y_i\in A^-$, $Y_i\in A_1^+$ implies that $Y_i\in A^+$. If $Y_i\in A_1^0$ then: $Y_i\in A^0$ iff $Y_i\cap U_0\neq\emptyset$, in case (i) we may choose $Y_i\in A^-$ or $Y_i\in A^+$, in case (ii) $Y_i\in A^-$, finally in case (iii) $Y_i\in A^+$.

If U_0 is $N(T)$-invariant then the choice in case (i) must be made in the following way: $Y_i\in A^-$ implies that $w(Y_i)\in A^+$. It is easy to check that U_0 is saturated in W and if U_0 is $N(T)$-invariant then W is also $N(T)$-invariant. This completes the proof of Lemma 11.

Now we come back to the proof of Lemma 10.

It follows from the above lemma that there exists a Weyl-invariant semi-section $A=(A^-,A^0,A^+)$ in X_1 such that $\Phi_{\mathcal{L}}(U)$ is saturated in $X_1^{\mathrm{ss}}(A)$. Thus $\Phi_{\mathcal{L}}(U)\subset X_1^{\mathrm{ss}}(A)\cap\Phi_{\mathcal{L}}(\bigcup_{g\in\mathrm{SL}(2)}gU)$. Moreover, since U is maximal in X with respect to the order $\triangleleft$, the set $\Phi_{\mathcal{L}}(U)$ is maximal with respect to $\triangleleft$ in $\Phi_{\mathcal{L}}(\bigcup_{g\in\mathrm{SL}(2)}gU)\subset X_1$.

Let $Z=(X_1^{\mathrm{ss}}(A)\cap\Phi_{\mathcal{L}}(\bigcup_{g\in\mathrm{SL}(2)}gU))-\Phi_{\mathcal{L}}(U)$. We want to show that for any $x\in Z$ either B^+x or B^-x is in Z.

Notice first that $x\in X_1^{\mathrm{ss}}(A)-X_1^{\mathrm{s}}(A)$ (if $x\in X_1^{\mathrm{s}}(A)\cap\Phi_{\mathcal{L}}(\bigcup_{g\in\mathrm{SL}(2)}gU)$, then by maximality of $\Phi_{\mathcal{L}}(U)$ in $\Phi_{\mathcal{L}}(\bigcup_{g\in\mathrm{SL}(2)}gU)$ we have $x\in\Phi_{\mathcal{L}}(U)$). Hence there exists $X_i\in A^0$ such that either $x\in X_i^+$ or $x\in X_i^-$. By symmetry we may assume that $x\in X_i^+$. Since $\Phi_{\mathcal{L}}(U)$ is open and saturated in $X_1^{\mathrm{ss}}(A)$ we have $X_i^+\cap\Phi_{\mathcal{L}}(U)=X_i\cap\Phi_{\mathcal{L}}(U)^+$. But for any $y\in X_i$, $\{y\}^+$ is B^+-invariant, hence $\{\lim_{t\to0}tx\}^+$ is B^+-invariant. Therefore $B^+x\cap\Phi_{\mathcal{L}}(U)=\emptyset$ and $B^+x\subset Z$.

Now we want to show that for any $x\in\Phi_{\mathcal{L}}(\bigcup_{g\in\mathrm{SL}(2)}gU)-X_1^{\mathrm{ss}}(A)$ either B^+x or B^-x is contained in Z. Since $\bigcup_{g\in\mathrm{SL}(2)}gU$ is SL(2)-invariant it suffices to show that either B^+x or B^-x is contained in $X_1-X_1^{\mathrm{ss}}(A)$. But this is clear since $X_1-X_1^{\mathrm{ss}}(A)=\bigcup_{j\in A^+}X_j^+\cup\bigcup_{j\in A^-}X_j^-$ and X_j^+,X_j^- are B^+- and B^--invariant, respectively.

It follows from the above results that for any $x\in\Phi_{\mathcal{L}}(\bigcup_{g\in\mathrm{SL}(2)}gU)-\Phi_{\mathcal{L}}(U)$ either B^+x or B^-x is contained in Z. Since $\Phi_{\mathcal{L}}$ is an SL(2)-invariant map, for any $x \in X - U$ either B^+x or B^-x is contained in $X - U$. Let F_1, F_2 be the sets of all $x \in X - U$ such that $B^+x \subset X - U$, $B^-x \subset X - U$, respectively. Then F_1, F_2 are obviously closed and $F_1 \cup F_2 = X$. The proof of Lemma 10 is complete.

Corollary 12. *Under the assumptions of Theorem* 9 *the set* $\bigcap_{g \in \mathrm{SL}(2)} gU$ *is open and* SL(2)*-invariant.*

Proof. In fact, since the sets $\mathrm{SL}(2)F_1$ and $\mathrm{SL}(2)F_2$ are B^+- and B^--invariant, respectively, and $\mathrm{SL}(2)/B^+$, $\mathrm{SL}(2)/B^-$ are complete we infer that $\mathrm{SL}(2)F_1$ and $\mathrm{SL}(2)F_2$ are closed. Hence $\bigcap_{g \in \mathrm{SL}(2)} gU = X - \mathrm{SL}(2)(X - U) = X - \mathrm{SL}(2)(F_1 \cup F_2)$ is open and obviously SL(2)-invariant.

Lemma 13. *Let* U *be an* $N(T)$*-invariant open subset of* X *such that* $X - U$ *is a union of* B^+*- and* B^-*-orbits and let* $x \in U$. *If* $\mathrm{SL}(2)x \cap (X - U) \neq \emptyset$, *then there exists* $b_1 \in B^+$ *such that* $b_1 x \in X - U$.

Proof. Let $\mathrm{SL}(2)x \cap (X - U) \neq \emptyset$. Then there exist $g_1, g_2 \in \mathrm{SL}(2)$ such that either $B^+g_1x \subset X - U$ or $B^-g_2x \subset X - U$. Assume that $B^+g_1X \subset X - U$. There exist $b_1, b_2 \in B^+$ such that $g = b_2\tau b_1$. Then also $\tau b_2 x \in X - U$. Since U is $N(T)$-invariant and $\tau \in N(T)$, we have $b_2x \in X - U$. If $B^-g_2x \subset X - U$, then we obtain $\tau B^-\tau^{-1}g_2x \subset X - U$, and hence $B^+(\tau^{-1}g_2)x \subset X - U$. Then, arguing as above for $g_1 = \tau^{-1}g_2$, we conclude that for some $b_2 \in B^+$, $b_2x \in x - U$.

Lemma 14. *Let* U *satisfy the assumptions of Lemma* 13. *Then* $\bigcap_{g \in \mathrm{SL}(2)} gU$ *is saturated in* U *with respect to the action of* T.

Proof. Let $x \in \bigcap_{g \in \mathrm{SL}(2)} gU$ and suppose that $y \in \overline{Tx} \cap U - Tx$. Then either $y = \lim_{t \to 0} tx$ or $y = \lim_{t \to \infty} tx$. Let $y = \lim_{t \to 0} tx$. Assume that $y \notin \bigcap_{g \in \mathrm{SL}(2)} gU$. Then $\mathrm{SL}(2)y \cap (X - U) \neq \emptyset$ and it follows from Lemma 13 that there exists $b_1 \in B^+$ such that $b_1y \in X - U$. But U is open and $y \in U^T$, hence $\{y\}^+ \subset U$. On the other hand, $B^+\{y\}^+ \subset \{y\}^+$. Thus $b_1y \in U$ and we have obtained a contradiction. This contradiction shows that $y \in \bigcap_{g \in \mathrm{SL}(2)} gU$. Thus $\bigcap_{g \in \mathrm{SL}(2)} gU$ is saturated in U with respect to the action of T.

Corollary 14. *Under the assumptions of Lemma* 13, *if there exists a good quotient* $U \to U//T$, *then there exists a good quotient* $\bigcap_{g \in \mathrm{SL}(2)} gU \to \bigcap_{g \in \mathrm{SL}(2)} gU//T$.

Proof of Theorem 9. Let U satisfy the assumptions of the theorem. It follows from Corollary 12 that U satisfies the assumptions of Lemma 13. Hence by Corollary 14, there exists a good quotient $\bigcap_{g \in \mathrm{SL}(2)} gU \to \bigcap_{g \in \mathrm{SL}(2)} gU//T$. By the Reduction Theorem (Theorem 5.1) of [BB-Ś4], we infer that there exists a good quotient $\bigcap_{g \in \mathrm{SL}(2)} gU \to \bigcap_{g \in \mathrm{SL}(2)} gU//\mathrm{SL}(2)$.

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