COLLOQUIUM MATHEMATICUM

VOL. LXIII 1992 FASC. 2

QUADRIC HYPERSURFACES OF FINITE TYPE

BY

BANG-YEN CHEN (EAST LANSING, MICHIGAN),
FRANKI DILLEN† (LEUVEN)
AND HONG-ZAO SONG‡ (HENAN)

Introduction. A submanifold M of the Euclidean m-space E^m is said to be of *finite type* (see [C1] for details) if each component of its position vector field X can be written as a finite sum of eigenfunctions of the Laplacian Δ of M, i.e., if

$$X = X_0 + X_1 + \ldots + X_k$$

where X_0 is a constant vector and $\Delta X_t = \lambda_t X_t$ for $t = 1, \ldots, k$. If in particular all eigenvalues $\lambda_1, \ldots, \lambda_k$ are mutually different, then M is said to be of *k-type*. If we define a polynomial P by

$$P(T) = \prod_{t=1}^{k} (T - \lambda_t)\,,$$

then $P(\Delta)(X - X_0) = 0$. If M is compact, then the converse also holds, i.e., if there exists a constant vector X_0 and a nontrivial polynomial P such that $P(\Delta)(X - X_0) = 0$, then M is of finite type [C1].

The class of finite type submanifolds is very large, including minimal submanifolds of E^m, minimal submanifolds of a hypersphere, parallel submanifolds, compact homogeneous submanifolds equivariantly immersed in a Euclidean space, and also isoparametric hypersurfaces of a hypersphere. On the other hand, very few hypersurfaces of finite type in a Euclidean space are known, other than minimal hypersurfaces (which are of 1-type). Therefore the following problem seems to be quite interesting.

PROBLEM. Classify all finite type hypersurfaces in E^m.

For $m = 2$, this problem was solved completely. In fact, it is known that

† Supported by a research fellowship of the Research Council of the Katholieke Universiteit Leuven.

‡ This work was done while the third author was a visiting scholar at Michigan State University. He would like to take this opportunity to express his hearty thanks to his colleagues there for their hospitality.

circles and straight lines are the only curves of finite type in E^2 (see [C1] and [CDVV] for details). For $m = 3$, the first result in this respect given in [C2], states that circular cylinders are the only tubes in E^3 which are of finite type. In [CDVV] it is shown that a ruled surface in E^3 is of finite type if and only if it is a plane, a circular cylinder or a helicoid. In [G], it is shown that a cone in E^m is of finite type if and only if it is minimal. In [D], some ruled submanifolds of finite type are classified.

If M' is an algebraic hypersurface with singularities in E^n, then M' is said to be of finite type if $M' - \{\text{singularities}\}$ is of finite type.

Combining the notion of algebraic hypersurfaces and the notion of submanifolds of finite type, the first two authors proved in [CD] that the only quadric surfaces of finite type in E^3 are the circular cylinders and the spheres. In this article, we shall completely classify quadric hypersurfaces of finite type.

2. Quadric hypersurfaces. A subset M of an n-dimensional Euclidean space E^n is called a *quadric hypersurface* if it is the set of points $(x_1, \ldots, x_n)$ satisfying the following equation of the second degree:

$$\sum_{i,k=1}^{n} a_{ik} x_i x_k + \sum_{i=1}^{n} b_i x_i + c = 0 \,, \tag{2.1}$$

where a_{ik}, b_i, c are all real numbers. We can assume without loss of generality that the matrix $A = (a_{ik})$ is symmetric and A is not a zero matrix. By applying a coordinate transformation in E^n if necessary, we may assume that (2.1) takes one of the following canonical forms:

$$\sum_{i=1}^{r} a_i x_i^2 + 1 = 0 \,, \tag{I}$$

$$\sum_{i=1}^{r} a_i x_i^2 + 2x_{r+1} = 0 \,, \tag{II}$$

$$\sum_{i=1}^{r} a_i x_i^2 = 0 \tag{III}$$

where $(a_1, \ldots, a_r, 0, \ldots, 0)$ (with $n - r$ zeros) is proportional to the eigenvalues of the matrix A. In general, we have $1 \le r \le n$. In the cases where $r = n$ in (I) and (III) and $r + 1 = n$ in (II) the hypersurface is called a *properly* $(n-1)$*-dimensional quadric hypersurface*, and in other cases, a *quadric cylindrical hypersurface*. In cases (I) and (III), the quadric cylindrical hypersurface is the product of an $(n - r)$-dimensional linear subspace E^{n-r} and a properly $(r - 1)$-dimensional quadric hypersurface. In case (II), the quadric cylindrical hypersurface is the product of an $(n-r-1)$-dimensional linear subspace and a properly r-dimensional quadric hypersurface.

Let $S^p(r)$ denote the hypersphere in E^{p+1} with radius r and centered at the origin. Denote by $M_{p,q}$ the product of spheres

$$S^p\left(\sqrt{\frac{p}{p+q}}\right) \times S^q\left(\sqrt{\frac{q}{p+q}}\right) \subset S^{p+q+1}(1) \subset E^{p+q+2}\,.$$

We denote by $C_{p,q}$ the $(p+q+1)$-dimensional cone in E^{p+q+2} with vertex at the origin shaped on $M_{p,q}$. It is easy to see that $C_{p,0}$ and $C_{0,q}$ are hyperplanes in E^{p+2} and E^{q+2}, respectively, and $C_{p,q}$ with $p > 0$, $q > 0$ are algebraic hypersurfaces of degree 2.

The purpose of this article is to prove the following classification theorem.

THEOREM. *A quadric hypersurface M in E^{n+1} is of finite type (even locally) if and only if it is one of the following hypersurfaces:*

(a) *hypersphere,*

(b) *one of the algebraic cones $C_{p,n-p-1}$, $0 < p < n-1$,*

(c) *the product of a linear subspace E^l and a hypersphere of E^{n-l+1} $(0 < l < n)$,*

(d) *the product of a linear subspace E^l and one of the algebraic cones $C_{p,n-l-p-1}$ $(0 < p < n-l-1)$.*

3. Properly n-dimensional quadric hypersurfaces. Let M be a hypersurface in E^{n+1}. Consider a parametrization

$$X(u_1,\ldots,u_n) = (u_1,\ldots,u_n,v) \tag{3.1}$$

where

$$v = v(u_1,\ldots,u_n)\,. \tag{3.2}$$

Denote $\partial_i v (= \partial v/\partial u_i)$ by v_i. Then we have

$$g_{ij} = \delta_{ij} + v_i v_j\,, \qquad g^{ij} = \delta_{ij} - \frac{v_i v_j}{g} \tag{3.3}$$

where

$$g = \det(g_{ij}) = 1 + \sum_{i=1}^{n} v_i^2\,, \tag{3.4}$$

and $g_{ij} = \langle \partial_i X, \partial_j X\rangle$. The Laplacian Δ of M is given by

$$\Delta = -\sum_{i,j}\left(\frac{\partial_i g}{2g} g^{ij} + \partial_i g^{ij}\right)\partial_j - \sum_{i,j} g^{ij}\partial_i\partial_j\,. \tag{3.5}$$

If M is a properly n-dimensional quadric hypersurface, then either M is an algebraic cone of degree 2 or M is of one of the following two kinds:

$$\text{(I)} \qquad v^2 = \sum_{i=1}^{n} b_i u_i^2 + c\,, \qquad b_1 \dots b_n c \neq 0\,,$$

$$\text{(II)} \qquad v = \frac{1}{2}\sum_{i=1}^{n} b_i u_i^2\,, \qquad b_1 \dots b_n \neq 0\,.$$

In the following two sections, we study properly n-dimensional quadric hypersurfaces of kinds (I) and (II), separately.

4. Proper quadric hypersurfaces of kind (I). In this section we assume M is a properly n-dimensional quadric hypersurface of kind (I). We may consider the following parametrization:

$$X = (u_1, \dots, u_n, v)\,, \quad v^2 = a_1 u_1^2 + \dots + a_n u_n^2 + c\,, \quad a_1 \dots a_n c \neq 0\,. \tag{4.1}$$

In this case, we have

$$v_i = \partial_i v = a_i u_i / v\,. \tag{4.2}$$

Thus, (3.3) and (3.4) imply

$$g_{ij} = \delta_{ij} + \frac{a_i a_j u_i u_j}{W}\,, \qquad g^{ij} = \delta_{ij} - \frac{a_i a_j u_i u_j}{gW}\,, \tag{4.3}$$

$$g = 1 + \frac{1}{W}\sum_i (a_i u_i)^2\,, \qquad \frac{1}{g} = 1 - \frac{1}{gW}\sum_i (a_i u_i)^2\,, \tag{4.4}$$

where

$$W = v^2 = a_1 u_1^2 + \dots + a_n u_n^2 + c\,. \tag{4.5}$$

From (4.4) we find

$$\partial_i g = \frac{2}{W}(a_i u_i (1 + a_i - g))\,, \tag{4.6}$$

$$\widetilde{g} := gW = c + \sum_i (1 + a_i) a_i u_i^2\,. \tag{4.7}$$

We put

$$\begin{aligned} A_k &= \frac{1}{2W}\Big\{(gW - a_k^2 u_k^2)\partial_k g - a_k u_k \sum_{t \neq k} a_t u_t \partial_t g\Big\} \\ &= \frac{1}{2} g \sum_t g^{tk} \partial_t g\,. \end{aligned} \tag{4.8}$$

Then from (4.3) and a straightforward computation, we have

$$-\sum_t \partial_t g^{tk} = \frac{a_k u_k}{gW}\sum_{t \neq k} a_t + \frac{2A_k}{g^2}\,. \tag{4.9}$$

From (3.5), (4.8) we obtain

$$\Delta = \frac{1}{g^2}\sum_i A_i\partial_i + \frac{1}{gW}\sum_j \Big(\sum_{t\neq k} a_t\Big) a_j u_j \partial_j - \sum_{i,j} g^{ij}\partial_i\partial_j\,. \tag{4.10}$$

We put

$$c_{ij} = g g^{ij}\,. \tag{4.11}$$

From (4.3), (4.4) and (4.11) we have

$$c_{ij} = \delta_{ij} + \frac{1}{W}\Big(\delta_{ij}\sum_t a_t^2 u_t^2 - a_i a_j u_i u_j\Big)\,. \tag{4.12}$$

For later use, we note that from (4.8), (4.12) we have

$$\sum_{i,j} c_{ij}(\partial_i g)(\partial_j g) = 2\sum_j A_j \partial_j g\,. \tag{4.13}$$

Also note from (4.7) that

$$\widetilde{g} = gW \text{ is a polynomial in } u_1,\ldots,u_n\,. \tag{4.14}$$

Lemma 1. *We have*

$$\Delta^t u_k = g^{1-3t} A_k \alpha_t \Big(\sum_i A_i \partial_i g\Big)^{t-1} + g^{2-3t} P_{k,t}(u_1,\ldots,u_n,1/W)$$

where $P_{k,t}$ is a polynomial in $n+1$ variables and α_t is given by

$$\alpha_t = (4-3t)(6t-5)\alpha_{t-1}\,, \qquad \alpha_1 = 1\,. \tag{4.15}$$

Proof. The proof goes by induction. For $t = 1$, the formula follows from (4.10). Suppose the lemma is true for $t-1$. Then it follows from (4.10), (4.11) and (4.13) that

$$\begin{aligned}\Delta^t u_k &= g^{1-3t}\sum_j A_j A_k \alpha_{t-1}\Big(\sum_i A_i\partial_i g\Big)^{t-2}(4-3t)\partial_j g\\ &\quad - g^{1-3t}\sum_{i,j} c_{ij} A_k \alpha_{t-1}\Big(\sum_l A_l \partial_l g\Big)^{t-2}(4-3t)(3-3t)\partial_j g\partial_i g\\ &\quad + g^{2-3t} P_{k,t}(u_1,\ldots,u_n,1/W)\\ &= g^{1-3t} A_k \alpha_t\Big(\sum_i A_i \partial_i g\Big)^{t-1} + g^{2-3t}P_{k,t}(u_1,\ldots,u_n,1/W)\,,\end{aligned}$$

which proves the lemma.

Now, suppose that M is of k-type. Then there exist real numbers $c_1,\ldots,c_k$ such that

$$\Delta^{k+1}X + c_1\Delta^k X + \ldots + c_k \Delta X = 0\,, \tag{4.16}$$

$$\Delta^{k+1}u_i + c_1\Delta^k u_i + \ldots + c_k\Delta u_i = 0\,, \qquad i = 1,\ldots,n\,. \tag{4.17}$$

From Lemma 1 and (4.17) we get

$$\Big(\sum_i A_i\partial_i g\Big)^{k+1} = gP(u_1,\dots,u_n,1/W)\,, \tag{4.18}$$

where P is a polynomial in $n+1$ variables. We put

$$G(u_1,\dots,u_n) = W^5\sum_i A_i\partial_i g\,. \tag{4.19}$$

Then G is a polynomial in $u_1,\dots,u_n$. Since W is a polynomial in $u_1,\dots,u_n$, there is a natural number N and a polynomial R in n variables such that

$$W^N P(u_1,\dots,u_n,1/W) = R(u_1,\dots,u_n)\,. \tag{4.20}$$

From (4.7), (4.18)–(4.20), we have

$$W^{N+1}G^{k+1} = \widetilde{g}W^{5k+5}R\,. \tag{4.21}$$

For any fixed j, $1\le j\le n$, we put $u_i=0$ for $i\ne j$ in (4.21) to obtain

$$\begin{aligned}(c+a_ju_j^2)^{N+k+2}2^{k+1}(a_j^2cu_j)^{2k+2}\\ = (c+a_j(a_j+1)u_j^2)(c+a_ju_j^2)^{5k+5}R(0,\dots,0,u_j,0,\dots,0)\,.\end{aligned} \tag{4.22}$$

Since $a_1\dots a_nc\ne 0$, this implies $a_j=-1$. Because this is true for any j, M is a hypersphere.

5. Proper quadric hypersurfaces of kind (II). For such hypersurfaces we consider a parametrization

$$X=(u_1,\dots,u_n,v)\,,\qquad v=\frac12\sum_i b_iu_i^2\,,\qquad b_1\dots b_n\ne 0\,. \tag{5.1}$$

From (3.3)–(3.5) we may find

$$g_{ij}=\delta_{ij}+b_ib_ju_iu_j\,,\qquad g^{ij}=\delta_{ij}-\frac{b_ib_ju_iu_j}{g}\,, \tag{5.2}$$

$$g=\det(g_{ij})=1+\sum_i b_i^2u_i^2\,, \tag{5.3}$$

$$\begin{aligned}\Delta=\frac{1}{g^2}\sum_j\Big\{b_j+\sum_i(b_j-b_i)b_i^2u_i^2\Big\}b_ju_j\partial_j\\ -\sum_{i,j}g^{ij}\partial_i\partial_j+\frac1g\sum_j\Big(\sum_{i\ne j}b_i\Big)b_ju_j\partial_j\,.\end{aligned} \tag{5.4}$$

Lemma 2. *We have*

$$g^2\Delta g = Q(u_1,\dots,u_n)+gT(u_1,\dots,u_n)\,, \tag{5.5}$$

$$||\nabla g||^2=\frac2g Q(u_1,\dots,u_n)\,, \tag{5.6}$$

where Q and T are some polynomials in $u_1, \ldots, u_n$ and ∇g is the gradient of g.

Proof. From (5.3) and (5.4) we find

$$\Delta g = \frac{2}{g^2} \sum_j b_j^2 u_j \Big\{ (b_j + \sum_i (b_j - b_i) b_i^2 u_i^2) b_j u_j + g \Big(\sum_{i \neq j} b_i \Big) b_j u_j \Big\} - 2 \sum_j b_j^2 g^{jj} .$$

Thus, if we put

$$Q = 2 \sum_j b_j^3 u_j^2 \Big\{ b_j + \sum_i (b_j - b_i) b_i^2 u_i^2 \Big\} , \tag{5.7}$$

$$T = 2 \sum_j b_j^3 u_j^2 \Big(\sum_{i \neq j} b_i \Big) - 2g \sum_i g^{ii} b_i^2 , \tag{5.8}$$

then we obtain (5.5). It is obvious that Q and T are polynomials in $u_1, \ldots, u_n$. (5.6) follows from the definition of the norm of ∇g, (5.2), (5.3) and (5.7).

Lemma 3. *We have*

$$\Delta^t u_j = g^{1-3t} Q^{t-1} b_j u_j \Big\{ b_j + \sum_i (b_j - b_i) b_i^2 u_i^2 \Big\} \alpha_t + g^{2-3t} \widetilde{P}_{j,t}$$

where $\widetilde{P}_{j,t}$ is a polynomial in $u_1, \ldots, u_n$ and α_t is given by (4.15).

Proof. The proof goes by induction. For $t = 1$ the formula follows easily from (5.4). Assume it is true for $t - 1$. Then we have

$$\begin{aligned} \Delta^t u_j &= \Delta \Big\{ g^{4-3t} Q^{t-2} b_j u_j \Big(b_j + \sum_i (b_j - b_i) b_i^2 u_i^2 \Big) \alpha_{t-1} + g^{5-3t} \widetilde{P}_{j,t-1} \Big\} \\ &= g^{1-3t} Q^{t-2} b_j u_j \Big(b_j + \sum_i (b_j - b_i) b_i^2 u_i^2 \Big) \alpha_{t-1} \\ &\quad \times \{ (4-3t) g^2 \Delta g - (4-3t)(3-3t) g ||\nabla g||^2 \} + g^{2-3t} \widehat{P}_{j,t} . \end{aligned}$$

where $\widehat{P}_{j,t}$ is a polynomial in $u_1, \ldots, u_n$. Thus, Lemma 2 implies the assertion.

If M is of k-type, then again there exist real numbers $c_1, \ldots, c_k$ such that

$$\Delta^{k+1} u_j + c_1 \Delta^k u_j + \ldots + c_k \Delta u_j = 0 , \quad j = 1, \ldots, n .$$

From Lemma 3 and (5.7) we obtain

$$Q^{k+1} = g P(u_1, \ldots, u_n)$$

where P is a polynomial in $u_1, \ldots, u_n$. Since $b_1 \ldots b_n \neq 0$, $g = 1 + \sum b_i^2 u_i^2$ is irreducible. Moreover, because $Q/g = \frac{1}{2} ||\nabla g||^2$ is not a polynomial in $u_1, \ldots, u_n$, we obtain a contradiction. Thus, there exist no proper quadric hypersurfaces of kind (II) which are of finite type.

6. Proof of Theorem. If M is a properly n-dimensional quadric hypersurface of finite type in E^{n+1}, then either M is an algebraic conic hypersurface of degree 2 or, according to §§3–5, M is a hypersphere. If M is an algebraic conic hypersurface of degree 2, then because M is of finite type, M is a minimal cone [G]. Thus, by a result of [H], M is one of the algebraic cones $C_{p,n-p-1}$, $0 < p < n-1$.

If M is a quadric cylindrical hypersurface of finite type in E^{n+1}, then M is the product of a linear subspace E^l and a proper quadric hypersurface, say N. Since M is of finite type, N is also of finite type. Thus, N is either a hypersphere or an algebraic cone $C_{p,n-l-p-1}$ for some suitable p.

The converse is easy to verify.

REFERENCES

[C1] B. Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific, Singapore 1984.

[C2] —, *Surfaces of finite type in Euclidean* 3*-space*, Bull. Soc. Math. Belg. Sér. B 39 (1987), 243–254.

[CD] B. Y. Chen and F. Dillen, *Quadrics of finite type*, J. Geom. 38 (1990), 16–22.

[CDVV] B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *Ruled surfaces of finite type*, Bull. Austral. Math. Soc. 42 (1990), 447–453.

[D] F. Dillen, *Ruled submanifolds of finite type*, Proc. Amer. Math. Soc. 114 (1992), 795–798.

[G] O. J. Garay, *Finite type cones shaped on spherical submanifolds*, ibid. 104 (1988), 868–870.

[H] W. Y. Hsiang, *Remarks on closed minimal submanifolds in the standard Riemannian m-sphere*, J. Differential Geom. 1 (1967), 257–267.

DEPARTMENT OF MATHEMATICS
MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN 48824-1027
U.S.A.

DEPARTEMENT WISKUNDE
KATHOLIEKE UNIVERSITEIT LEUVEN
CELESTIJNENLAAN 200B
B-3001 LEUVEN, BELGIUM

DEPARTMENT OF MATHEMATICS
HENAN UNIVERSITY
KAIFENG, HENAN 475001
PEOPLE'S REPUBLIC OF CHINA

Reçu par la Rédaction le 30.8.1990