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ON K-CONTACT RIEMANNIAN MANIFOLDS WITH VANISHING E-CONTACT BOCHNER CURVATURE TENSOR

BY

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1. Introduction. For Sasakian manifolds, Matsumoto and Chūman [6] defined the contact Bochner curvature tensor (see also Yano [9]). Hasegawa and Nakane [4] and Ikawa and Kon [5] have studied Sasakian manifolds with vanishing contact Bochner curvature tensor. Such manifolds were studied in the theory of submanifolds by Yano ([9] and [10]).

In this paper we define an extended contact Bochner curvature tensor in K-contact Riemannian manifolds and call it the E-contact Bochner curvature tensor. Then we show that a K-contact Riemannian manifold with vanishing E-contact Bochner curvature tensor is a Sasakian manifold.

2. Preliminaries. Let M be a $(2n+1)$-dimensional contact metric manifold with the structure tensor (ϕ,ξ,η,g). Then

$$\phi\xi=0\,,\quad \eta(\xi)=1\,,\quad \phi^2=-I+\eta\otimes\xi\,,\quad \eta(X)=g(\xi\,,X)\,,$$
$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\quad g(\phi X,Y)=d\eta(X,Y)$$

for any vector fields X and Y on M. If ξ is a Killing vector field on a contact metric manifold M, M is said to be a *K-contact Riemannian manifold*. In a K-contact Riemannian manifold, we have

$$\nabla_X\xi=\phi X,\quad (\nabla_X\phi)Y=R(X,\xi)Y\,,\quad R(X,\xi)\xi=-\eta(X)\xi+X \tag{2.1}$$

(see e.g. [3] and [11]),

$$g(Q\xi,\xi)=2n \tag{2.2}$$

(see e.g. [1] and [2]) and

$$(\nabla_{\phi X}\phi)\phi Y+(\nabla_X\phi)Y=-2g(X,Y)\xi+\eta(Y)(X+\eta(X)\xi) \tag{2.3}$$

(see [7]), where ∇ is the covariant differentiation with respect to g, R is the Riemannian curvature tensor of M and Q is the Ricci operator of M. If a contact metric manifold M is normal (i.e., $N+d\eta\otimes\xi=0$, where N denotes the Nijenhuis tensor formed with ϕ), M is called a *Sasakian manifold*. It is well known that in a Sasakian manifold with structure tensors (ϕ,ξ,η,g)

we have

$$(\nabla_X \phi)Y = R(X,\xi)Y = -g(X,Y)\xi + \eta(Y)X\,.$$

Matsumoto and Chūman [6] defined the *contact Bochner curvature tensor* B of a Sasakian manifold M $(m = 2n)$ by

$$\begin{aligned}(2.4)\qquad B(X,Y) = R(X,Y) &+ \frac{1}{m+4}(QY \wedge X - QX \wedge Y + Q\phi Y \wedge \phi X \\ &- Q\phi X \wedge \phi Y + 2g(Q\phi X, Y)\phi \\ &+ 2g(\phi X, Y)Q\phi + \eta(Y)QX \wedge \xi + \eta(X)\xi \wedge QY) \\ &- \frac{k+m}{m+4}(\phi Y \wedge \phi X + 2g(\phi X, Y)\phi) \\ &- \frac{k-4}{m+4} Y \wedge X + \frac{k}{m+4}(\eta(Y)\xi \wedge X + \eta(X)Y \wedge \xi)\,,\end{aligned}$$

where $k = (S+m)/(m+2)$ (S is the scalar curvature of M) and $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$. The tensor B is invariant with respect to a D-homothetic deformation

$$*g = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta\,, \quad *\phi = \phi\,, \quad *\xi = \alpha^{-1}\xi\,, \quad *\eta = \alpha\eta\,,$$

where α is a positive constant.

From now on for a D-homothetic deformation we say that $M(\phi,\xi,\eta,g)$ is *D-homothetic* to $M(*\phi,*\xi,*\eta,*g)$. It is well known that if a contact metric manifold $M(\phi,\xi,\eta,g)$ is D-homothetic to $M(*\phi,*\xi,*\eta,*g)$, then $M(*\phi,*\xi,*\eta,*g)$ is a contact metric manifold. Moreover, if $M(\phi,\xi,\eta,g)$ is a K-contact Riemannian manifold (resp. Sasakian manifold), then $M(*\phi,*\xi,*\eta,*g)$ is also a K-contact Riemannian manifold (resp. Sasakian manifold) (see [8]).

Now we define the *extended contact Bochner curvature tensor* on a K-contact Riemannian manifold M by

$$\begin{aligned}&B^e(X,Y)Z \\ &\quad = B(X,Y)Z - \eta(X)B(\xi,Y)Z - \eta(Y)B(X,\xi)Z - \eta(Z)B(X,Y)\xi\end{aligned}$$

for any vector fields X, Y and Z, B being formally defined as in (2.4). We call B^e the *E-contact Bochner curvature tensor*. In particular, if M is a Sasakian manifold, we have $B(\xi,Y)Z = B(Y,Z)\xi = 0$. Thus B^e coincides with B on a Sasakian manifold M. Moreover, by using (2.1)–(2.3), one can show that a D-homothetic deformation on a K-contact Riemannian manifold M satisfies the following equations (see also Tanno [8]):

$$\begin{aligned}*R(X,Y)Z = R(X,Y)Z &\\ &+ (\alpha-1)(2g(\phi X,Y)\phi Z + g(\phi Z,Y)\phi X - g(\phi Z,X)\phi Y) \\ &+ (\alpha-1)(\eta(Y)R(X,\xi)Z + \eta(Z)R(X,\xi)Y - \eta(X)R(Y,\xi)Z \\ &- \eta(Z)R(Y,\xi)X) - (\alpha-1)^2(\eta(Z)\eta(X)Y - \eta(Y)\eta(Z)X)\,,\end{aligned}$$

$$\begin{aligned} *\,\mathrm{Ric}(X,Y) = {} & \mathrm{Ric}(X,Y) - 2(\alpha-1)g(X,Y) + 2(2n+1)(\alpha-1)\eta(X)\eta(Y) \\ & + 2n(\alpha-1)^2\eta(X)\eta(Y), \end{aligned}$$

$$\begin{aligned} (g(*QX,Y) = {} & \frac{1}{\alpha}g(QX,Y) - \frac{2(\alpha-1)}{\alpha}g(X,Y) \\ & + \frac{2(2n+1)(\alpha-1)}{\alpha}\eta(X)\eta(Y) - \frac{\alpha-1}{\alpha}\eta(Y)g(Q\xi,X)), \end{aligned}$$

$$*S = \frac{1}{\alpha}S - \frac{2n(\alpha-1)}{\alpha},$$

$$\begin{aligned} *B(X,Y)Z = {} & B(X,Y)Z + (\alpha-1)(-\eta(Y)(-g(X,Z)\xi + \eta(Z)X) \\ & - \eta(Z)(-g(X,Y)\xi + \eta(Y)X) + \eta(X)(-g(Y,Z)\xi + \eta(Z)Y) \\ & + \eta(Z)(-g(Y,X)\xi + \eta(X)Y) + \eta(Y)R(X,\xi)Z \\ & + \eta(Z)R(X,\xi)Y - \eta(X)R(Y,\xi)Z - \eta(Z)R(Y,\xi)X), \end{aligned}$$

where Ric is the Ricci curvature tensor of M. Thus, after some lengthy computation (using also (2.1)) we can see that $*B^e = B^e$ on a K-contact Riemannian manifold.

3. Results. We define

$$S^{\#} = \sum_{i,j=1}^{2n+1} g(R(E_i,E_j)\phi E_j, \phi E_i), \tag{3.1}$$

where $\{E_i\}$ is an orthonormal frame.

LEMMA ([7]). *For any* $(2n+1)$*-dimensional* K*-contact Riemannian manifold* M, *we have*

$$S^{\#} - S + 4n^2 = \frac{1}{2}\{\|\nabla\phi\|^2 - 4n\} \geq 0,$$

Moreover, M *is Sasakian if and only if* $\|\nabla\phi\|^2 = 4n$ *or equivalently* $S^{\#} - S + 4n^2 = 0$.

THEOREM. *Let* M *be a* K*-contact Riemannian manifold with vanishing* E*-contact Bochner curvature tensor. Then* M *is a Sasakian manifold.*

Proof. Since B^e vanishes, we have

$$\begin{aligned} & g(R(X,Y)Z,W) \\ & = -\frac{1}{m+4}(g(X,Z)g(QY,W) - g(QY,Z)g(X,W) \\ & \quad - g(Y,Z)g(QX,W) + g(QX,Z)g(Y,W) \\ & \quad + g(\phi X,Z)g(Q\phi Y,W) - g(Q\phi Y,Z)g(\phi X,W) \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&- g(\phi Y, Z)g(Q\phi X, W) + g(Q\phi X, Z)g(\phi Y, W) \\
&+ 2g(Q\phi X, Y)g(\phi Z, W) + 2g(\phi X, Y)g(Q\phi Z, W) \\
&+ \eta(Y)(g(\xi, Z)g(QX, W) - g(QX, Z)g(\xi, W)) \\
&+ \eta(X)(g(QY, Z)g(\xi, W) - g(\xi, Z)g(QY, W))) \\
&+ \frac{k+m}{m+4}(g(\phi X, Z)g(\phi Y, W) - g(\phi Y, Z)g(\phi X, W) \\
&+ 2g(\phi X, Y)g(\phi Z, W)) \\
&+ \frac{k-4}{m+4}(g(X, Z)g(Y, W) - g(Y, Z)g(X, W)) \\
&- \frac{k}{m+4}(\eta(Y)(g(X, Z)g(\xi, W) \\
&- g(\xi, Z)g(X, W)) + \eta(X)(g(\xi, Z)g(Y, W) \\
&- g(Y, Z)g(\xi, W))) + \eta(X)g(B(\xi, Y)Z, W) \\
&+ \eta(Y)g(B(X, \xi)Z, W) + \eta(Z)g(B(X, Y)\xi, W)\,.
\end{aligned}
$$

Using (3.1) and (3.2) we find

$$
\begin{aligned}
S^{\#} &= \sum_{i,j=1}^{2n+1} g(R(E_i, E_j)\phi E_j, \phi E_i) \\
&= \frac{4(n+1)}{m+4}(S - g(Q\xi, \xi)) - \frac{2n+k}{m+4}(4n^2 + 2n) - \frac{2n(k-4)}{m+4}\,.
\end{aligned}
$$

Hence, using $m = 2n$, $k = (S+2n)/(2n+2)$ and (2.2), we get $S^{\#} = S - 4n^2$. By the Lemma our result follows.

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REFERENCES

[1] D. E. Blair, *On the non-existence of flat contact metric structures*, Tôhoku Math. J. 28 (1976), 373–379.

[2] —, *Two remarks on contact metric structures*, ibid. 29 (1977), 319–324.

[3] H. Endo, *Invariant submanifolds in a K-contact Riemannian manifold*, Tensor (N.S.) 28 (1974), 154-156.

[4] I. Hasegawa and T. Nakane, *On Sasakian manifolds with vanishing contact Bochner curvature tensor II*, Hokkaido Math. J. 11 (1982), 44–51.

[5] T. Ikawa and M. Kon, *Sasakian manifolds with vanishing contact Bochner curvature tensor and constant scalar curvature*, Colloq. Math. 37 (1977), 113–122.

[6] M. Matsumoto and G. Chūman, *On the C-Bochner tensor*, TRU Math. 5 (1969), 21–30.

[7] Z. Olszak, *On contact metric manifolds*, Tôhoku Math. J. 31 (1979), 247–253.

[8] S. Tanno, *The topology of contact Riemannian manifolds*, Illinois J. Math. 12 (1968), 700–712.

[9] K. Yano, *Anti-invariant submanifolds of a Sasakian manifold with vanishing contact Bochner curvature tensor*, J. Differential Geom. 12 (1977), 153–170.
[10] —, *Differential geometry of anti-invariant submanifolds of a Sasakian manifold*, Boll. Un. Mat. Ital. 12 (1975), 279–296.
[11] K. Yano and M. Kon, *Structures on Manifolds*, World Scientific, Singapore 1984.

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