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VOL. LXII 1991 FASC. 2

A NOTE ON COUNTABLE CONNECTED LOCALLY CONNECTED URYSOHN ALMOST REGULAR SPACES

BY

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In [5] G. X. Ritter poses the question whether there exists a countable, connected, locally connected, Urysohn, almost regular space. This question has been answered in the affirmative in [1], [2] and [4].

In this note we give another example of such a space, using a different method: Starting from an arbitrary countable, connected, locally connected, Urysohn space ([3], [5]), we construct a new countable, connected, locally connected, Urysohn space S extendable to a countable, connected, locally connected, Urysohn and almost regular space S^*. This extension is constructed by adjoining a specific (extending) family of countable open non-accumulating filterbases on S.

A topological space X is called *Urysohn* if any two distinct points of X have disjoint closed neighbourhoods, and *almost regular* if there exists a dense subset of X at every point of which the space X is regular.

A) *The space S.* Let T be a countable, connected, locally connected Urysohn space and let p be a fixed point of T.

To every point $t \in T$ we attach a copy R_t of the space $R = T \setminus \{p\}$, identifying the point t with p.

In the set

$$I^1(T) = T \cup \bigcup_{t\in T} R_t$$

we define the following topology: The points in R_t, $t \in T$, keep their own bases of open neighbourhoods. For a point $t \in T$, let $B(t)$ denote a basis of open neighbourhoods of t in T. If we set $B^-(p) = \{H = U \setminus \{p\} : U \in B(p)\}$ and $B^-_t(p)$ is the copy of $B^-(p)$ in R_t, then a basis of open neighbourhoods of $t \in T$ in $I^1(T)$ is

$$B^1(t) = \Big\{U \cup \bigcup_{s\in U\setminus\{t\}} R_s \cup H_t : U \in B(t)\,,\ H_t \in B^-_t(p)\Big\}\,.$$

It can be easily proved that $I^1(T)$ is countable, connected, locally connected and Urysohn.

In the same manner we construct the spaces $I^1(R_t) = R_t \cup \bigcup_{s \in R_t} R_s$, and in the set $I^2(T) = T \cup \bigcup_{t \in T} I^1(R_t)$ we define a topology in exactly the same manner as in $I^1(T)$.

By induction, we construct the spaces $I^3(T), \ldots, I^n(T), \ldots$, and we consider the set $S = \bigcup_{n=1}^{\infty} I^n(T)$. Observe that in the set J consisting of the initial space T and of all copies of $R = T \setminus \{p\}$ attached to the spaces T, $I^1(T), \ldots, I^n(T), \ldots$, the relation "$L \leq M$ if M is attached to L" is a partial order such that the set of all predecessors of any element of J is well-ordered by $\leq$. Hence $(J, \leq)$ is a tree whose first level is the initial space T.

In S we define the following topology: $U \subseteq S$ is basic open in S if $U \cap I^n(T)$ is open in $I^n(T)$ for every $n = 1, 2, \ldots$, and there exists a natural number m such that for every $s \in U \cap (I^{m+1}(T) \setminus I^m(T))$ the tree whose first level is the copy R_s attached to s, is included in U. It can be easily proved that S is countable, connected, locally connected and Urysohn.

B) *The space* S^*. Let G be the set of all countable open non-accumulating filterbases of S "generated" by chains (i.e. well-ordered subsets) meeting every level of J. Thus, if $p \in G, p = \{p_k\}, k = 1, 2, \ldots$, then each p_k is an open set of S identified with the tree whose first level is the copy $p_k \cap I^{n+k-1}(T)$ attached to a point $t_{k-1} \in I^{n+k-2}(T)$. (For $n = k = 1$ we set $I^0(T) = T$.) Hence, $\mathrm{cl}_S\, p_k = p_k \cup \{t_{k-1}\}$, $p_{k+1} \subseteq \mathrm{cl}_S\, p_{k+1} \subseteq p_k$ for every $k = 1, 2, \ldots$, and $\bigcap_{k=1}^{\infty} p_k = \emptyset$.

Let G^* be the subset of G such that if $p = \{p_k\}$, $k = 1, 2, \ldots$, then the points t_{k-1}, $k = 1, 2, \ldots$, to which the sets p_k are attached correspond to a constant sequence in T.

Consider the set $S^* = S \cup G^*$. It can be easily proved that if U is an open subset of S and $U^* = U \cup \{p \in G^* : U \text{ includes a member of } p\}$, then $B = \{U^* : U \text{ open in } S\}$ is a basis for a topology in S^*. By the definition of G^* it follows that S^* is Hausdorff (for, if $p, q \in G^*$ and $p \neq q$ then there exist $p_k \in p$, $q_n \in q$ such that $p_k \cap q_n = \emptyset$, and if $p \in G^*$ and $x \in S$ then there exist $p_k \in p$ and an open neighbourhood U of x such that $p_k \cap U = \emptyset$).

Obviously, for every open set U^* of S^*, we have

$$\mathrm{cl}_{S^*}\, U^* = \mathrm{cl}_S\, U \cup (U^* \setminus U)\,. \tag{a}$$

Proposition. *The space* S^* *is countable*, *connected*, *locally connected*, *Urysohn and almost regular.*

Proof. S^* is countable because G^* is countable; connected, because S is connected and dense in S^*; locally connected, because if U is open connected in S, then U^* is open connected in S^* (since U is dense in U^*); Urysohn, because S is Urysohn and from (a), if $x \in \mathrm{cl}_{S^*}\, U^* \setminus U^*$ then $x \in \mathrm{cl}_S\, U \setminus U$; regular at every $p \in S^* \setminus S$, because if U^* is an open set containing p, then there exists a natural number k such that $p_{k+1} \subseteq \mathrm{cl}_S\, p_{k+1} \subseteq U$, which implies $p \in p^*_{k+1} \subseteq \mathrm{cl}_{S^*} p^*_{k+1} \subseteq U^*$ (since, from (a), $\mathrm{cl}_{S^*} p_{k+1} = \mathrm{cl}_S p_{k+1} \cup (p^*_{k+1} \setminus p_{k+1})$). Finally, S^* is almost regular, because for every open subset U^* of S^*, $U^* \cap (S^* \setminus S) \neq \emptyset$, that is, $S^* \setminus S$ is dense in S^*.

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Reçu par la Rédaction le 28.8.1989;
en version modifiée le 30.8.1990