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REMARKS ON FINITE REGULAR PLANES

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In this communication I shall present some remarks on the classification of finite regular planes (see [3]) from the point of view of the notion of dimension. Moreover, I shall present the constructions of some A_n^3-algebras (see problems **P 374** and **P 375** in [3]) and some observations on the existence of non-isomorphic algebras.

Let $P = \langle X, R_3 \rangle$ be a *finite regular plane*, i. e. X is a set consisting of finitely many elements (points) and R_3 is a relation of three arguments being points of X, for which the following conditions hold:

A1. If $a = b$, then $R_3(a, b, c)$.

A2. If $R_3(a, b, c)$, then $R_3(b, a, c)$ and $R_3(c, a, b)$.

A3. If $a \neq b$, $R_3(a, b, c)$ and $R_3(a, b, d)$, then $R_3(b, c, d)$.

A4. There exist points $a, b, c \in X$ such that $\sim R_3(a, b, c)$.

A5. All straight lines (1) consist of the same number of points.

Let $\sim R_3(a, b, c)$. Now I define by induction a sequence of sets $[a, b, c]_0, [a, b, c]_1, \ldots, [a, b, c]_n, \ldots$ such that

1° $[a, b, c]_0 = \{a, b, c\}$,

2° $x \in [a, b, c]_{k+1}$ if and only if there exist points, $d, e \in [a, b, c]_k$, $d \neq e$ such that $R_3(d, e, x)$.

It is easy to see that $[a, b, c]_p \subseteq [a, b, c]_q$ if $p \leqslant q$.

The set $[a, b, c] = \bigcup_{n=1}^{\infty} [a, b, c]_n$ is called the *full subset of X generated by the points* $a, b, c \in X$ ($P^* = \langle [a, b, c]; R_3 \rangle$ is evidently a subplane of the plane $P = \langle X, R_3 \rangle$).

A plane $\langle X, R_3 \rangle$ will be called an *essential plane* if the following condition holds:

AW. *For every* $a, b, c \in X$ *if* $\sim R_3(a, b, c)$; *then* $[a, b, c] = X$.

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(1) *Straight line* means here: set of all those $x \in X$ for which $R_3(a, b, x)$ holds for fixed $a \neq b$ of X.

A plane $\langle X, R_3\rangle$ will be called a degenerated plane if

AW$_1$. *There exist points* $a, b, c, d, e, f \in X$, *such that* $\sim R_3(a, b, c)$, $\sim R_3(d, e, f)$, $[a, b, c] = X$ *and* $X \setminus [d, e, f] \neq 0$.

It is natural to call $P = \langle X, R_3\rangle$ a *finite space*, if it satisfies the following condition:

AW$_2$. *For every* $a, b, c \in X$ *if* $\sim R_3(a, b, c)$; *then* $X \setminus [a, b, c] \neq 0$.

The essential planes are A_7^3- and A_9^3-algebras. The A_{15}^3-algebra constructed in [3] (see the remarks in § 4) is a degenerated plane. The A_n^3-algebras whose construction will be presented in § 2 of this article, are finite spaces.

1. It is easy to prove that the essential planes have the following property by which their two-dimensionality is characterized.

THEOREM 1.1. *Let* $\langle X, R_3\rangle$ *be any essential plane, and* $R_4(x_1, x_2, x_3, x_4)$ *a relation for which the following conditions hold*:

B1. *If* $R_3(x_1, x_2, x_3)$, *then* $R_4(x_1, x_2, x_3, y)$,

B2. *If* $R_4(x_1, x_2, x_3, x_4)$, *then* $R_4(x_4, x_1, x_2, x_3)$ *and* $R_4(x_2, x_1, x_3, x_4)$,

B3. *If* $\sim R_3(x_1, x_2, x_3)$, $R_4(x_1, x_2, x_3, y)$ *and* $R_4(x_1, x_2, x_3, z)$, *then* $R_4(x_1, x_2, y, z)$.

Then for every $x_1, x_2, x_3, x_4 \in X$ *we have* $R_4(x_1, x_2, x_3, x_4)$.

The following lemmas are evident

L1. *If* $\sim R_3(a, b, c)$, $x \neq y$, $R_3(a, x, y)$ *and* $R_3(b, x, y)$, *then* $\sim R_3(c, x, y)$

L2. *If* $\sim R_3(a, b, c)$, $a \neq x$ *and* $R_3(a, b, x)$, *then* $\sim R_3(a, c, x)$.

Now we prove by induction

L3. *If* $\sim R_3(x_1, x_2, x_3)$ *and* $z \in [x_1, x_2, x_3]_n$, *then* $R_4(x_1, x_2, x_3, z)$.

In fact, if $z \in [x_1, x_2, x_3]_0$, then z is one of the points x_1, x_2, x_3, thus $R_4(x_1, x_2, x_3, z)$ according to A1, B1, B2. From B1 and B2 it follows that $R_4(x_1, x_2, x_3, z)$ for $z \in [x_1, x_2, x_3]_1$ as well.

Assume now that

(*) *for every* $a \in [x_1, x_2, x_3]_k$ $(k > 0)$ *it is* $R_4(x_1, x_2, x_3, a)$.

Let $z \in [x_1, x_2, x_3]_{k+1}$. According to the definition there exist such $a_1, a_2 \in [x_1, x_2, x_3]_k$ that $R_3(a_1, a_2, z)$ and $a_1 \neq a_2$, whence we obtain, making use of B1,

(i) $$R_4(x_i, a_1, a_2, z) \quad \text{for} \quad i = 1, 2, 3.$$

From (*) we obtain $R_4(x_1, x_2, x_3, a_j)$ for $j = 1, 2$. Now, because of $\sim R_3(x_1, x_2, x_3)$ we have, by virtue of B3,

(ii) $$R_4(x_i, x_j, a_1, a_2) \quad \text{for} \quad i, j = 1, 2, 3.$$

By virtue of L1 we may (without loss of generality) assume that

(iii) $$\sim R_3(x_1, a_1, a_2).$$

In view of B3 we have from (i) and (ii)

(iv) $$R_4(x_1, x_i, a_j, z) \quad \text{for} \quad i = 2, 3;\ j = 1, 2.$$

According to L2 we have $R_3(x_1, a_1, z)$ or $\sim R_3(x_1, a_2, z)$. For each of these alternatives we obtain, in virtue of B3 and (iv), $R_4(x_1, x_2, x_3, z)$. This completes the proof of L3.

Proof of theorem 1.1. Suppose that there exist x_1, x_2, x_3, x_4 for which $\sim R_4(x_1, x_2, x_3, x_4)$ holds. From B1 we obtain $\sim R_3(x_1, x_2, x_3)$ and, from AW, $x_4 \in [x_1, x_2, x_3]$, which in view of L3 contradicts the assumption. Thus $R_4(x_1, x_2, x_3, x_4)$ for every x_1, x_2, x_3, x_4.

2. In [3] I introduced the notion of A_n^3-algebras and showed their correspondence with P_i^3-planes $(n = 2i+1)$ for $i \geqslant 3$.

Namely, I call an *A_n^3-algebra* the ordered pair $\langle A, \circ\rangle$ where A is a set consisting of n points and $\circ$ is an operation having the following properties:

W1. $a \circ a = a$,

W2. $a \circ b = b \circ a$,

W3. $a \circ (a \circ b) = b$.

For these algebras the relation $R_3(a, b, c)$ (satisfied if and only if $a = b$ or $a = c$ or $b = c$ or $a \circ b = c$) was introduced in a natural manner.

An A_n^3-algebra satisfying the condition

W4. If $\sim R_3(a, b, c)$, then $a \circ (b \circ c) = (a \circ b) \circ c$ is called a *PA_n^3-algebra* (*projective A_n^3-algebra*).

I shall prove the following

THEOREM 2.1. *Given the algebra* $PA_n^3 = \langle A, \circ\rangle$, let $\{g_1, g_2, \ldots, g_s\} \subset A$ *be an arbitrary minimal set of generators of this algebra. The number of points of this algebra is an element of the sequence defined by formulae*: $u_1 = 1$, $u_s = 2u_{s-1}+1$ (*hence it is of the form* 2^s-1).

Proof. We shall consider the sequence of subalgebras $A_{l_p}^3 = \langle A_p, \circ\rangle$ of the given PA_n^3 $(p = 1, 2, \ldots, s)$, each of them generated by the subsequence $g_1, g_2, \ldots, g_p$ of its generators.

From W1 it follows, that $A_{l_1}^3$ has only one point. It is easy to see, that the number of elements of $A_{l_2} = \langle A_2, \circ\rangle$ is $l_2 = 3 = 2\cdot 1+1$.

Let $A_{l_p}^3 = \langle A_p, \circ\rangle$ be the subalgebra of the PA_n^3 generated by the elements $g_1, g_2, \ldots, g_p$. Let $A_p = \{a_1, a_2, \ldots, a_k\}$, $l_p = k$. We shall prove that the number of elements of the subalgebra $A_{l_{p+1}}^3$ generated by $g_1, g_2, \ldots, g_{p+1}$ is $l_{p+1} = 2\cdot k+1$.

From W2 and W3 we have

(1) If $a \circ b = a$, then $a = b$,

(2) If $a \circ b = a \circ c$, then $b = c$,

whence $a_i \circ g_{p+1} \neq a_j \circ g_{p+1}$ for $i \neq j$. Of course, $g_{p+1} \neq a_i$ and $a_i \circ g_{p+1} \neq g_{p+1}$. We write $\gamma_i = a_i \circ g_{p+1} = g_{p+1} \circ a_i$. Evidently $\gamma_i \neq a_i$.

We shall show that $A_{p+1} = \{a_1, \ldots, a_k, g_{p+1}, \gamma_1, \ldots, \gamma_k\}$.

Really

$$a_i \circ a_j = a_m \in A_{p+1},$$

$$a_i \circ g_{p+1} = g_{p+1} \circ a_i = \gamma_i \in A_{p+1},$$

$$\gamma_i \circ \gamma_i = \gamma_i \in A_{p+1},$$

$$g_{p+1} \circ g_{p+1} = g_{p+1} \in A_{p+1},$$

$$\gamma_i \circ a_i = a_i \circ \gamma_i = a_i \circ (a_i \circ g_{p+1}) = g_{p+1} \in A_{p+1},$$

$$\gamma_j \circ a_i = a_i \circ \gamma_j = a_i \circ (a_j \circ g_{p+1}) = (a_i \circ a_j) \circ g_{p+1} = a_m \circ g_{p+1} = \gamma_m \in A_{p+1},$$

$$\gamma_j \circ \gamma_i = \gamma_i \circ \gamma_j = \gamma_i \circ (a_j \circ g_{p+1}) = (\gamma_i \circ a_j) \circ g_{p+1} = (a_j \circ \gamma_i) \circ g_{p+1} = a_j \circ (\gamma_i \circ g_{p+1}) = a_j \circ [(a_i \circ g_{p+1}) \circ g_{p+1}] = a_j \circ a_i = a_m \in A_{p+1}.$$

This completes the proof of theorem 2.1.

We remark that for a fixed n all PA_n^3-algebras are isomorphic.

By constructing a sequence of algebras corresponding to the sequence given in [3] it is easy to prove that for each $n \geqslant 15$ there exist algebras, being not PA_n^3-algebras such that the number of their points is an element of the sequence $u_m = 3, 7, 15, 31, \ldots$

Any A_n^3-algebra for which the following condition holds

W4″. $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$,

will be called *AA_n^3-algebra (affine A_n^3-algebra)*.

I shall prove

THEOREM 2.2. *If $\{g_1, g_2, \ldots, g_s\}$ is an arbitrary minimal set of generators of AA_n^3-algebra, then the number of elements of this algebra is $n = 3^{s-1}$.*

Proof. Analogously as in the proof of theorem 2.1 we shall consider the sequence of subalgebras $A_{l_p}^3 = \langle A_p, \circ \rangle$ of this algebra with the same property as in 2.1.

It is easy to see, that the number of elements of $A_{l_2} = \langle A_2, \circ \rangle$ is $l_2 = 3 = 3^1$.

Let $A_{l_p}^3 = \langle A_p, \circ \rangle$ be the subalgebra of the AA_n^3 generated by $g_1, g_2, \ldots, g_p$. Let $A_p = \{a_1, a_2, \ldots, a_k\}$, $l_p = k$.

We shall prove that the number of elements of the $A_{l_{p+1}}^3$-algebra generated by $g_1, g_2, \ldots, g_{p+1}$ is $l_{p+1} = 3k$.

Of course $g_{p+1} \notin A_p$. From (2) we have $a_i \circ g_{p+1} \neq a_j \circ g_{p+1}$ for $i \neq j$.

We write $\gamma_i = a_i \circ g_{p+1} = g_{p+1} \circ a_i$ and $\delta_i = a_i \circ \gamma_1 = \gamma_1 \circ a_i$. Evidently $\gamma_i \neq a_j$ and $\delta_i \neq a_j$. Now we show that $\gamma_i \neq \delta_j$.

Let $a_j \circ a_i = a_k$, whence $a_j = a_i \circ a_k$. We have $\delta_j = a_j \circ \gamma_1 = (a_i \circ a_k) \circ (a_1 \circ g_{p+1}) = (a_k \circ a_1) \circ (a_i \circ g_{p+1}) = (a_k \circ a_1) \circ \gamma_i \neq \gamma_i$, because of (1) and $\gamma_i \neq a_s = a_k \circ a_1$.

We now prove that $A_{p+1} = \{a_1, \ldots, a_k, \gamma_1, \ldots, \gamma_k, \delta_1, \ldots, \delta_k\}$. Really $a_i \circ a_j = a_m \in A_{p+1}$; let $a_1 \circ a_i = a_k$, whence $a_i = a_1 \circ a_k$, and let $a_k \circ a_j = a_s$, then

$$\gamma_j \circ a_i = a_i \circ \gamma_j = (a_1 \circ a_k) \circ (a_j \circ g_{p+1}) = (a_k \circ a_j) \circ (a_1 \circ g_{p+1}) = a_s \circ \gamma_1 = \delta_s \in A_{p+1},$$

$$\delta_j \circ a_i = a_i \circ \delta_j = a_i \circ (a_j \circ \gamma_1) = (a_1 \circ a_k) \circ (a_j \circ \gamma_1) = (a_k \circ a_j) \circ (a_1 \circ \gamma_1) = a_s \circ [a_1 \circ (a_1 \circ g_{p+1})] = a_s \circ g_{p+1} = \gamma_s \in A_{p+1},$$

$$\gamma_k \circ \gamma_j = (a_k \circ g_{p+1}) \circ (a_j \circ g_{p+1}) = g_{p+1} \circ (a_k \circ a_j) = g_{p+1} \circ a_s = \gamma_s \in A_{p+1},$$

$$\delta_k \circ \delta_j = (a_k \circ \gamma_1) \circ (a_j \circ \gamma_1) = (a_k \circ a_j) \circ \gamma_1 = a_s \circ \gamma_1 = \delta_s \in A_{p+1}.$$

Besides, let $a_s \circ a_1 = a_q$

$$\delta_j \circ \gamma_k = \gamma_k \circ \delta_j = (a_k \circ g_{p+1}) \circ (a_j \circ \gamma_1) = (a_k \circ a_j) \circ [g_{p+1} \circ (g_{p+1} \circ a_1)] = a_s \circ a_1 = a_q \in A_{p+1},$$

which completes the proof.

For any fixed n all AA_n^3-algebras are evidently isomorphic.

3. It seems to be interesting whether there exist, for a fixed n, different A_n^3-algebras which are essential planes, degenerated planes, or spaces (**P 388**). This problem is connected with a question raised by Skolem in [2] about the number (for fixed i) of the non-isomorphic planes P_i^3.

Evidently all A_7^3-algebras are isomorphic and are essential planes. The same can be said of A_9^3-algebras. All algebras constructed by the author in this article and in [3] are not essential planes for $n > 9$. The question can be raised whether there exist L_m-essential planes for $m > 2$. The answer is positive as will be proved now.

Let $P_i^k = \langle X, R_3 \rangle$, $k > 2$ (i is here the order of ramification of every point from X, and k is the number of points on every straight line) and $\sim R_3(a, b, c)$ for a certain triple $a, b, c \in X$.

The following lemmas are evident:

L1. *The number of elements of the set $[a, b, c]_1$ is $3(k-1)$.*

L2. *If $[a, b, c]_m = [a, b, c]$, then $m \geqslant 2$.*

L3. *The number of elements of the set* $[a, b, c]_2$ *is at least* $3(k-1)+$ $+(k-2)^2$.

Really, each of the straight lines $[a, b]$, $[a, c]$, $[b, c]$ contains exactly $k-2$ points different from a, b, c. They are all points of the set $[a, b, c]_1$. Let the points of one of these straight lines, e. g. $[a, b]$, be the points $a, b, x_1, x_2, \ldots, x_{k-2}$. On each of the straight lines $[c, x_1]$, $[c, x_2], \ldots,$ $[c, x_{k-2}]$ are placed at least $k-2$ points different from the points of the set $[a, b, c]_1$.

We now prove

THEOREM 3.1. *If* $P_i^k = \langle X, R_3 \rangle$ *is a degenerated plane or space, then the number of the points of* X *is at least equal to* $(k-1)[3(k-1)+(k-2)^2]+1$ (*thus* $i \geqslant 3(k-1)+(k-2)^2$).

Proof. Taking into account the condition AW_1 or the condition AW_2 we infer that there exists a triple $a, b, c \in X$ for which $X \setminus [a, b, c]$ $\neq 0$. From L2 it follows that if $[a, b, c]_m = [a, b, c]$; then $m \geqslant 2$. Let $x \in X$ and $x \notin [a, b, c]_2$. Let $x_1, x_2, \ldots, x_l$ be points of the set $[a, b, c]_3$. From L3 we have $l \geqslant 3(k-1)+(k-2)^2$. From the definition of the sequence of the sets $[a, b, c]_n$ it follows that the points of the straight lines $[x, x_p]$ different from x_p do not belong to $[a, b, c]_2$. There are at least $(k-2)l$ such points. The number of points of the plane P_i^k is at least equal to $3(k-1)+(k-2)^2+(k-2)[3(k-1)+(k-2)^2]+1 =$ $=(k-1)[3(k-1)+(k-2)^2]+1$.

Therefore all planes $P_i^k = \langle X, R_3 \rangle$ for which $i < 3(k-1)+(k-2)^2$ are essential planes.

The A_{13}^3-algebra (P_6^3-planes or L_3-planes)

○	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	8	9	10	11	12	13	2	3	4	5	6	7
2	8	2	10	11	12	13	9	1	7	3	4	5	6
3	9	10	3	12	13	8	11	6	1	2	7	4	5
4	10	11	12	4	8	7	6	5	13	1	2	3	9
5	11	12	13	8	5	9	10	4	6	7	1	2	3
6	12	13	8	7	9	6	4	3	5	11	10	1	2
7	13	9	11	6	10	4	7	12	2	5	3	8	1
8	2	1	6	5	4	3	12	8	10	9	13	7	11
9	3	7	1	13	6	5	2	10	9	8	12	11	4
10	4	3	2	1	7	11	5	9	8	10	6	13	12
11	5	4	7	2	1	10	3	13	12	6	11	9	8
12	6	5	4	3	2	1	8	7	11	13	9	12	10
13	7	6	5	9	3	2	1	11	4	12	8	10	13

is an essential plane.

It is easy to verify that the following A_{15}^3-algebra is also an essential plane:

○	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	9	10	11	12	13	14	15	2	3	4	5	6	7	8
2	9	2	11	12	13	14	15	10	1	8	3	4	5	6	7
3	10	11	3	13	14	15	8	7	12	1	2	9	4	5	6
4	11	12	13	4	15	7	6	9	8	14	1	2	3	10	5
5	12	13	14	15	5	8	9	6	7	11	10	1	2	3	4
6	13	14	15	7	8	6	4	5	10	9	12	11	1	2	3
7	14	15	8	6	9	4	7	3	5	12	13	10	11	1	2
8	15	10	7	9	6	5	3	8	4	2	14	13	12	11	1
9	2	1	12	8	7	10	5	4	9	6	15	3	14	13	11
10	3	8	1	14	11	9	12	2	6	10	5	7	15	4	13
11	4	3	2	1	10	12	13	14	15	5	11	6	7	8	9
12	5	4	9	2	1	11	10	13	3	7	6	12	8	15	14
13	6	5	4	3	2	1	11	12	14	15	7	8	13	9	10
14	7	6	5	10	3	2	1	11	13	4	8	15	9	14	12
15	8	7	6	5	4	3	2	1	11	13	9	14	10	12	15

Hence there exist at least 3 non-isomorphic A_{15}^3-algebras.

The author does not think that the problem raised above as Skolem's question in [2] may by solved in general, because the brilliant results of Skolem solving the problem of existence of the triple systems of Steiner touch the limit of the possibilities of contemporary mathematics.

Certain partial results concerning the existence and classification of A_n^3-algebras may by obtained by machine computation.

This situation may by compared with the results obtained in [1] where the problem of the existence of orthogonal Latin squares was solved. Here the problem of existence of projective and affine planes in general seems to be unsolvable, and partial results were obtained by machine computation.

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