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MAPPINGS OF INVERSE LIMITS

BY

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The purpose of this note is to give necessary and sufficient conditions for a compact metric space be a continuous image of another one expressed in terms of inverse expansions in polyhedra. Also conditions for homeomorphism are given. These are analogous to the conditions given by Alexandroff [1] and Švedov [5] for another kind of inverse expansions. The results of this note have applications in [4].

1. Preliminaries. We consider inverse limits in the sense of [2]. Let $X = \varprojlim\{X_n, \pi_n^m, M\}$, where $n, m \in M$, M is a directed set, $\pi_n^m: X_m \to X_n$, $m \geqslant n$, are continuous mappings [1] and X_n are polyhedra. We denote by π_n projections from X into X_n. Let $Y = \varprojlim\{Y_n, \sigma_n^m, N\}$ be another such a limit. Let $f: X \to Y$ be a mapping. We shall use the following approximation lemma of [2] (Theorem X. 11. 9 with Lemmas X. 3.7 and X. 3.8) which may be expressed, for the case considered here, as follows:

LEMMA 1. *For every $n \in N$ and $\varepsilon > 0$ there exists $m_0 \in M$ such that for every $m \geqslant m_0$ there exists a mapping $f_{mn}: X_m \to Y_n$ such that the diagram*

$$\begin{array}{ccc} X_m & \leftarrow & X \\ \downarrow & & \downarrow \\ Y_n & \leftarrow & Y \end{array} \tag{1}$$

is ε-commutative, i. e. the distance between $f_{mn}\pi_m(x)$ and $\sigma_n f(x)$ is less than ε for every $x \in X$.

2. Mappings of compact metric spaces. We consider now inverse systems of polyhedra $\{X_n, \pi_n^m\}$, where π_n^m are onto and m, n are positive integers. According to Freudenthal [3], every compact metric space is an inverse limit of such a system.

THEOREM 1. *If a space $Y = \varprojlim\{Y_n, \sigma_n^m\}$ is a continuous image of*

(1) Throughout this note all mappings are assumed to be continuous.

a space $X = \lim\limits_{\leftarrow}\{X_n, \pi_n^m\}$, *then for every sequence* $\{\varepsilon_n\}$, *where* $\varepsilon_n > 0$ *and* $\lim \varepsilon_n = 0$, *there exists an infinite diagram*

$$\begin{array}{ccccccccc} X_{m_1} & \leftarrow & X_{m_2} & \leftarrow \dots \leftarrow & X_{m_k} & \leftarrow & X_{m_{k+1}} & \leftarrow \dots \\ f_1\downarrow & & f_2\downarrow & & f_k\downarrow & & f_{k+1}\downarrow & \\ Y_{n_1} & \leftarrow & Y_{n_2} & \leftarrow \dots \leftarrow & Y_{n_k} & \leftarrow & Y_{n_{k+1}} & \leftarrow \dots, \end{array} \tag{2}$$

where $\{m_k\}$ *and* $\{n_k\}$ *are non-decreasing and unbounded sequences of positive integers, and every subdiagram of the form*

$$\begin{array}{ccccc} & & X_{m_k} & \leftarrow & X_{m_r} \\ & & \downarrow & & \downarrow \\ Y_{n_i} & \leftarrow & Y_{n_k} & \leftarrow & Y_{n_r} \end{array} \tag{3}$$

is ε_k*-commutative for all* $i \leqslant k$ *and* $r \geqslant k$.

Proof. Let $Y = f(X)$. We define the required diagram by induction. Let $n_1 = 1$. According to Lemma 1, we choose m_1 and $f_1: X_{m_1} \to Y_{n_1}$ such that diagram (1) is ε_1-commutative for $m = m_1$, $n = n_1$ and $f_{mn} = f_1$.

Suppose that m_k, n_k and f_k are already defined for $k \leqslant j$ and that they have the following properties:

1° subdiagrams (3) lying in the already constructed part of (2) are ε_k-commutative,

2° diagrams (1) for $n = n_k$, $m = m_k$ and $f_{mn} = f_k$ are ε_k-commutative.

Let $m_{j+1} \geqslant n_j$. Note first that there exists $\eta_0 > 0$ such that η-commutativity of the diagram (1) for $\eta \leqslant \eta_0$, $n = n_{j+1}$ and $m \geqslant m_j$ implies ε_k-commutativity of diagrams (3) with $m_r = m_k$, $n_r = n_{j+1}$ and $k \leqslant j$. Choose $\eta \leqslant \min(\eta_0, \varepsilon_{j+1})$ and then choose $m_{j+1} \geqslant m_j$ and $f_{j+1}: X_{m_{j+1}} \to Y_{n_{j+1}}$ such that the diagram (1) be η-commutative for $n = n_{j+1}$, $m = m_{j+1}$ and $f_{mn} = f_{j+1}$. It easy to verify that the properties 1° and 2° hold for $k \leqslant j+1$.

Remark. If Y is of dimension 0, then diagrams of Theorem 1 may be taken simply commutative. It would be interesting to known whether it is possible to obtain the commutativity in Theorem 1 for the dimension of Y greater than 0 (**P 389**).

A special case of the following theorem is known from [2] (Theorem VIII.3.13). It gives a sufficient condition for a compact metric space be a continuous image of another one. We shall write $f \underset{\varepsilon}{=} g$ if the distance between $f(x)$ and $g(x)$ is less than ε for all x.

THEOREM 2. *Let* $\{\varepsilon_n\}$, $n = 1, 2, \dots$, *be a sequence of positive numbers such that* $\lim \varepsilon_n = 0$. *The existence of an infinite diagram* (2) *having, with respect to this sequence, the properties required in Theorem* 1 *induces the existence of continuous mapping* $f: X \to Y$ *such that* $\sigma_s^{n_k} f_k \pi_{m_k} \underset{\varepsilon_k}{=} \sigma_s f$ *for every* s *and* k, $s \leqslant n_k$, *and which is onto if* f_k *are onto.*

Proof. We define f as a mapping which sends $x = (x_1, x_2, \dots)$ onto $y = (y_1, y_2, \dots)$, where $y_s = \lim\limits_{k\to\infty} \sigma_s^{n_k} f_k(x_{m_k})$, $s = 1, 2, \dots$ The limit exists because for $r \geqslant k$ we have $\sigma_s^{n_k} f_k \pi_{m_k} \underset{\varepsilon_k}{=} \sigma_s^{n_r} f_r \pi_{m_r}$, according to ε_k-commutativity of diagram (3). The point y defined in this way belongs to Y, as

$$\sigma_i^j(y_j) = \sigma_i^j[\lim_{k\to\infty} \sigma_j^{n_k} f_k(x_{m_k})] = \lim_{k\to\infty} \sigma_i^j \sigma_j^{n_k} f_k(x_{m_k})$$
$$= \lim_{k\to\infty} \sigma_i^{n_k} f_k(x_{m_k}) = y_i, \quad i = 1, 2, \dots$$

The ε_k-equalities required by the Theorem are valid according to the definition of f.

In order to prove the continuity of f, it is sufficient to prove the continuity of $\sigma_s f$ for every $s = 1, 2, \dots$ Let $\varepsilon > 0$ and s be given. Choose k such that $\sigma_s^{n_k} f_k \pi_{m_k} \underset{\varepsilon}{=} \sigma_s f$. Let x' and $x'' \in X$ be such that the distance between $\sigma_s^{n_k} f_k(x'_{m_k})$ and $\sigma_s^{n_k} f_k(x''_{m_k})$ is not greather than ε. Then the distance between $\sigma_s f(x')$ and $\sigma_s f(x'')$ is not greather than 3ε. The continuity is proved.

Now assume that f_k are onto. Let $y = (y_1, y_2, \dots) \in Y$. As f_k are onto, then $(\sigma_s^{n_k} f_k \pi_{m_k})^{-1}(y_s)$ are non-empty sets for every s and k, $s \leqslant n_k$. We denote by A_s the topological limit superior of these sets if $k \to \infty$. It is easy to verify that if $x \in A_s$, then $f(x) = y_s$. Note that $A_s \subset A_t$ for $t \geqslant s$. Let then $x \in \bigcap_{s=1}^{\infty} A_s$. Then $\sigma_s f(x) = y_s$ for every s, i. e. $f(x) = y$.

More convenient for applications is the following weaker form of the above Theorem. Let $\mathcal{F}$ be a class of mappings.

THEOREM 2′. *If for every pair of positive integers* m *and* n, *for every mapping* $f_{mn}: X_m \to Y_n$ *belonging to* $\mathcal{F}$, *and for every* $\varepsilon > 0$ *and* $n' > n$ *there exist* $m' > m$ *and a mapping* $f_{m'n'}: X_{m'} \to Y_{n'}$ *belonging to* $\mathcal{F}$ *such that the diagram*

$$\begin{array}{ccc} X_m & \leftarrow & X_{m'} \\ \downarrow & & \downarrow \\ Y_n & \leftarrow & Y_{n'} \end{array} \tag{4}$$

is ε*-commutative, then there exists a mapping* $f: X \to Y$ *which is onto if all mappings in* $\mathcal{F}$ *are onto.*

The proof reduces to the verification that the hypotheses of Theorem 2′ implies the hypotheses of Theorem 2, i. e. to the construction of diagram (2). This construction is made by induction which is standard and therefore will be omitted.

3. Homeomorphism of compact metric spaces. We prove now

THEOREM 3. *If* $X = \varprojlim\{X_n, \pi_n^m\}$ *and* $Y = \varprojlim\{Y_n, \sigma_n^m\}$ *are homeomorphic, then for every sequence* $\{\varepsilon_n\}$ *such that* $\varepsilon_n > 0$ *and* $\lim \varepsilon_n = 0$, *there exists an infinite diagram*

$$(5)\qquad \begin{array}{ccccccccc} X_{m_1} & \leftarrow & X_{m_2} & \leftarrow \ldots \leftarrow & X_{m_{2k-1}} & \leftarrow & X_{m_{2k}} & \leftarrow \ldots \\ \downarrow & & \uparrow & & \downarrow & & \uparrow & \\ Y_{n_1} & \leftarrow & Y_{n_2} & \leftarrow \ldots \leftarrow & Y_{n_{2k-1}} & \leftarrow & Y_{n_{2k}} & \leftarrow \ldots, \end{array}$$

where $\{m_k\}$ *and* $\{n_k\}$ *are unbounded and non-decreasing sequences of positive integers, and every subdiagram of the form*

$$(5')\qquad \begin{array}{ccccc} & & X_{m_{2k-1}} & \leftarrow & X_{m_{2r}} \\ & & \downarrow & & \uparrow \\ Y_{n_i} & \leftarrow & Y_{n_{2k-1}} & \leftarrow & Y_{n_{2r}} \end{array} \qquad\qquad (5'')\qquad \begin{array}{ccccc} X_{m_i} & \leftarrow & X_{m_{2k}} & \leftarrow & X_{m_{2r-1}} \\ & & \uparrow & & \downarrow \\ & & Y_{n_{2k}} & \leftarrow & Y_{n_{2r-1}} \end{array}$$

$$(5''')\qquad \begin{array}{ccccc} X_{m_i} & \leftarrow & X_{m_{2k}} & \leftarrow & X_{m_{2r}} \\ & & \uparrow & & \uparrow \\ & & Y_{n_{2k}} & \leftarrow & Y_{n_{2r}} \end{array} \qquad\qquad (5'''')\qquad \begin{array}{ccccc} & & X_{m_{2k-1}} & \leftarrow & X_{m_{2r-1}} \\ & & \downarrow & & \downarrow \\ Y_{n_i} & \leftarrow & Y_{n_{2k-1}} & \leftarrow & Y_{n_{2r-1}} \end{array}$$

is ε_{2k}-*commutative in the cases* $(5'')$ *and* $(5''')$ *and* ε_{2k-1}-*commutative in the cases* $(5')$ *and* $(5'''')$.

Proof. Let $Y = f(X)$ and $X = g(Y)$, where fg and gf are identities. We construct the required diagram by induction. Let $n_1 = 1$. According to Lemma 1, we choose m_1 and $f_1\colon X_{m_1} \to Y_{n_1}$ such that diagram (1) is ε_1-commutative for $m = m_1$, $n = n_1$ and $f_{mn} = f_1$. Suppose that m_{2k}, n_{2k}, m_{2k-1}, n_{2k-1}, f_k and g_k are already defined for indices $2k-1$ and $2k$ not greather than $j = 2p-1$ (the case $j = 2p$ is symmetric to this one) and that they have the following properties:

1° subdiagrams (5′)-(5″) lying in the already constructed part of (5) are ε_{2k}-commutative and ε_{2k-1}-commutative respectively,

2° diagrams (1) for $f_{mn} = f_k$ are ε_{2k-1}-commutative and similar diagrams for g_k and g are ε_{2k}-commutative.

Let $m_{2p} \geqslant m_{2p-1}$. Note that there exist $\eta_0 > 0$ such that η-commutativity of the diagram of type (1) for $\eta \leqslant \eta_0$, for mapping g instead of f and for mapping $g_{n,m_{2p}}$ instead of f_{mn}, where $n \geqslant n_{2p-1}$, implies the ε_{2k}-commutativity and ε_{2k-1}-commutativity of diagrams (5′) and (5‴) for $n_{2r} = n$, $m_{2r} = m_{2p}$ and all k such that $2k$ and $2k-1$ is not greater than $j = 2p-1$. We choose $\eta \leqslant \min(\eta_0, \varepsilon_{2p})$ and then we choose $n_{2p} \geqslant n_{2p-1}$ and $g_p\colon Y_{n_{2p}} \to X_{m_{2p}}$ such that the diagram of the type (1) for g and g_p is η-commutative. It is easy to verify that the properties 1° and 2° hold for $2k$ and $2k-1$ not greather than $2p$.

Remark. As in the case of Theorem 1, diagram (4) may be taken simply commutative if X and Y are of dimension 0.

THEOREM 4. *Let* $\{\varepsilon_n\}$, $n = 1, 2, \ldots$, *be a sequence of positive numbers such that* $\lim \varepsilon_n = 0$. *The existence of an infinite diagram* (4) *having, with respect to this sequence, the properties required in Theorem* 3 *induces the existence of a homeomorphism* f *of* X *onto* Y (*the inverse of* f *is denoted by* g) *such that* $\sigma_s^{n_{2k-1}} f_k \pi_{m_{2k-1}} \underset{\varepsilon_{2k-1}}{=} \sigma_s f$ *and* $\pi_s^{m_{2k}} g_k \sigma_{n_{2k}} \underset{\varepsilon_{2k}}{=} \pi_s g$ *for every* s *and* k, $s \leqslant n_{2k-1}$ *in the first case, and* $s \leqslant m_{2k}$ *in the second one.*

Proof. According to Theorem 2, diagram (4) induces the existence of mappings $f\colon X \to Y$ and $g\colon Y \to X$. It remains to show that fg and gf are identities. We shall verify only the first inequality. According to the definition of f and g (see the proof of Theorem 2) we have

$$\begin{aligned} \sigma_s fg(y) &= \lim_{k\to\infty} \sigma_s^{n_{2k-1}} f_k [\lim_{r\to\infty} \pi_{m_{2k-1}}^{m_{2r}} g_r(y_{n_{2r}})] \\ &= \lim_{k\to\infty}\lim_{r\to\infty} \sigma_s^{n_{2k-1}} f_k \pi_{m_{2k-1}}^{m_{2r}} g_r(y_{n_{2r}}) \\ &= \lim_{k\to\infty} \sigma_s^{n_{2k-1}} (y'_{n_{2k-1}}), \end{aligned}$$

where $y'_{n_{2k-1}} \underset{\varepsilon_{2k-1}}{=} y_{n_{2k-1}}$ by ε_{2k-1}-commutativity of diagram (5′). By commutativity (6′), we have also $\sigma_s^{n_{2k-1}}(y'_{n_{2k-1}}) \underset{\varepsilon_{2k-1}}{=} y_s$. Hence, we have $\lim_{k\to\infty} \sigma_s^{n_{2k-1}}(y'_n) = y_s$. Thus, $\sigma_s fg(y) = y$ for every s, i. e. fg is the identity.

The more convenient for applications is the following weaker form of the above Theorem. Let $\mathscr{F}$ and $\mathscr{G}$ be classes of mappings.

THEOREM 4′. *If for every pair of positive integers* m *and* n, *for every mapping* $f_{mn}\colon X_m \to Y_n$ *belonging to* $\mathscr{F}$, *for every* $\varepsilon > 0$ *and* $m' > m$, *there exists* $n' > n$ *and a mapping* $g_{n'm'}\colon Y_{n'} \to X_{m'}$ *belonging to* $\mathscr{G}$ *such that the diagram*

$$\begin{array}{ccc} X_m & \leftarrow & X_{m'} \\ \downarrow & & \uparrow \\ Y_n & \leftarrow & Y_{n'} \end{array}$$

is ε-*commutative, and the same is true after change* X *into* Y, $\mathscr{F}$ *into* $\mathscr{G}$ *etc., then there exists a homeomorphism between* X *and* Y.

The proof reduces to the verification that the hypotheses of Theorem 4′ implies hypotheses of Theorem 4, i. e. to the construction of diagram (4). The construction is made by induction which is standard and therefore will be omitted.

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ON MAPPINGS THAT CHANGE DIMENSIONS OF SPHERES

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A mapping (i.e. a continuous function) f of the space X is said to be *strongly irreducible* provided that $f(A) = f(X)$ implies $A = X$ for every closed subset A of X (see [3], p. 162).

We denote by D_f the set of all points of X on which f is 1-1, i.e.

$$D_f = \{x : x \in X,\, x = f^{-1}f(x)\};$$

the mapping f is obviously strongly irreducible if the set D_f is dense in X. It is known that the inverse is also true provided that X is a compact metric space (see [3], p. 163).

Examples. Let us denote by bI^n the boundary of the n-dimensional cube I^n in the n-dimensional Euclidean space E^n.

There exists a strongly irreducible monotone mapping f of I^n ($n = 2, 3, \ldots$) such that $f(bI^n)$ is a point and $\dim f(I^n) = 1$ (1). *Hence $f(I^n)$ is a dendrite* (see [1], p. 333, 336 and 338).

Indeed, let C be the Cantor ternary set on the segment $I = \{t : 0 \leqslant t \leqslant 1\}$, and let P be an arbitrary n-dimensional parallelepiped in E^n, with boundary bP and centre q. Consider a set A, consisting of points $q + c(x-q)$, where $c \in C - \{1\}$, $x \in bP$, and E^n is understood to be a vector space. Then A is a nowhere dense closed subset of $P - bP$, and each component B of $(P-bP)-A$ is a domain in E^n, bounded by surfaces $q + c_i(bP-q)$, clearly homeomorphical to bP, where $i = 1, 2$, and c_1, c_2 are end points of a component interval of $I - C$. Let us cut every domain B with compact pieces of $(n-1)$-dimensional hyperplanes contained in the closure of B into a finite number of parallelepipeds P' whose diameters $\delta(P')$ are less than $\frac{1}{2}\delta(P)$. Denote by A' the union of A and of all these $(n-1)$-dimensional pieces, where B ranges over the countable collection of all components of $(P-bP)-A$. So A' is also closed in $P - bP$. Hence the collection $\boldsymbol{C}(P)$ of components of A' is one of continua

(1) The idea of the example is due to K. Sieklucki.