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ON THE DIMENSION OF SEMI-COMPACT SPACES AND THEIR QUASI-COMPONENTS

BY

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All spaces under discussion will be separable and metrizable. X is called *semi-compact* if each point of X has arbitrarily small neighborhoods with compact boundary.

In [4] A. Lelek has proved the following interesting theorem:

THEOREM. *Suppose X has the following three properties:*

(1) *each quasi-component of X is locally compact;*

(2) *each quasi-component of X is of dimension $\leqslant 0$;*

(3) *X is semi-compact.*

Then the dimension of X is $\leqslant 0$.

In the same paper, A. Lelek posed the following question ([4], P 373, p. 244): Is the above theorem true if zero is replaced by n? In the present paper, we prove a theorem (Theorem 2) which has the above theorem as a special case and gives an affirmative answer to the above question*.

1. A lemma on embedding. In this section, we prove a lemma which leads to a special embedding of a space X into a subspace Z of an appropriate compact space.

Given a space X, there is a continuous function f defined on X into C, where C is the Cantor ternary set, such that $\{f^{-1}(y) \mid y \in f(X)\}$ is the collection of all quasi-components of X ([3], p. 93). This function f gives rise to a family $\mathcal{U}$ of simultaneously open and closed subsets of X defibers ned as follows:

$$\mathcal{U} = \{U \mid U = f^{-1}(G),\ G \text{ is both open and closed in } f(X)\}.$$

$\mathcal{U}$ has the property that the intersection of a finite number of members of $\mathcal{U}$ is again a member of $\mathcal{U}$.

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LEMMA 1. *Let X be given and f be the function defined above. Suppose Y is a compactification of X and $I = [0, 1]$. Then there is a subspace Z of $I \times Y$ with the following properties:*

(1) *Z is the union of a family of mutually disjoint compact sets Z_Q indexed by the quasi-components Q of X;*

(2) *for each quasi-component Q of X, $Z_Q = \{f(Q)\} \times \tilde{Q}$, where $Q \subset \tilde{Q}$ and $\tilde{Q} \setminus Q \subset Y \setminus X$;*

(3) *for each Z_Q, there are arbitrarily small simultaneously open and closed neighborhoods of Z_Q in Z.*

Proof. Let $\mathscr{U} = \{U\}$ be the family of simultaneously open and closed subsets U of X defined above. Since U is closed in X, $\overline{U} \cap X = U$, where $\overline{U}$ is the closure of U in Y. For each quasi-component Q of X, we have a collection $\mathscr{U}_Q = \{U \in \mathscr{U} \mid Q \subset U\}$. Q is the intersection of all U, $U \in \mathscr{U}_Q$. Let $\tilde{Q} = \bigcap \overline{U}$ where the intersection is taken over all $U \in \mathscr{U}_Q$. Clearly, $\tilde{Q}$ is compact, $Q \subset \tilde{Q}$, $(X \setminus Q) \cap Q = \emptyset$ and $\tilde{Q} \setminus Q \subset Y \setminus X$.

For each quasi-component Q of X, let $Z_Q = \{f(Q)\} \times \tilde{Q}$ and $Z = \bigcup Z_Q$, where the union is taken over all quasi-components Q of X. Clearly, (1) and (2) are satisfied by Z. We need only verify (3).

Let W be an open neighborhood in $I \times Y$ of Z_Q. Since Z_Q is compact and $Z_Q = \{f(Q)\} \times \tilde{Q}$, we may assume $W = (\alpha, \beta) \times V$, where (α, β) is an open interval containing $f(Q)$ and V is an open set in Y containing $\tilde{Q}$. By the Lindelöf theorem, there is a countable collection $U_m \in \mathscr{U}_Q$ such that $\tilde{Q} = \bigcap_{m=1}^{\infty} \overline{U}_m$. Let $\tilde{V}_m = \bigcap_{i=1}^{m} \overline{U}_i$. Then,

$$\tilde{V}_m \cap X = (\bigcap_{i=1}^{m} \overline{U}_i) \cap X = \bigcap_{i=1}^{m} U_i \in \mathscr{U}_Q.$$

Let $V_m = \bigcap_{i=1}^{m} U_i$. Then, $\tilde{Q} \subset \overline{V}_m \subset \tilde{V}_m$. Hence, $\bigcap_{m=1}^{\infty} \overline{V}_m = \tilde{Q}$. Since $\{\overline{V}_m\}$ is nested and V is an open neighborhood of $\tilde{Q}$, there is an n_0 such that $\overline{V}_{n_0} \subset V$. Let Q' be any quasi-component of X such that $Q' \subset V_{n_0}$. Then $\tilde{Q}' \subset \overline{V}_{n_0} \subset V$. Since $f(V_{n_0})$ is both open and closed in $f(X)$, there is an open interval $(\overline{\alpha}, \overline{\beta})$ such that $f(Q) \in (\overline{\alpha}, \overline{\beta}) \subset (\alpha, \beta)$, $f^{-1}((\overline{\alpha}, \overline{\beta})) \subset V_{n_0}$ and $(\overline{\alpha}, \overline{\beta}) \cap f(X)$ is both open and closed in $f(X)$. Consider the neighborhood $W' = (\overline{\alpha}, \overline{\beta}) \times V$ of Z_Q. Obviously, $W' \subset W$. If T is the union of all $Z_{Q'}$ such that $\overline{\alpha} < f(Q') < \overline{\beta}$, then $T \supset Z \cap W'$ and T is both open and closed in Z. Consider any $Z_{Q'}$ contained in T. Then, $\overline{\alpha} < f(Q') < \overline{\beta}$. Hence $\tilde{Q}' \subset V$. Consequently, $Z_{Q'} = \{f(Q')\} \times \tilde{Q}' \subset (\overline{\alpha}, \overline{\beta}) \times V = W'$. Therefore, $T = W' \cap Z$. This completes the proof of (3). The proof of lemma 1 is now complete.

It is clear that the graph of f is contained in Z. Since the graph of f is homeomorphic with X, we have the following lemma:

LEMMA 2. *Let X and Z be as in lemma* 1. *If* $\dim Z_Q \leqslant n$ *for all quasi-components Q of X, then* $n \geqslant \dim Z \geqslant \dim X$.

Proof. The collection of sets $\{Z_Q\}$ has the properties that $\dim Z_Q \leqslant n$ and each Z_Q has arbitrarily small neighborhoods with empty boundary in Z. Hence, by [2], proposition G, p. 90, $\dim Z \leqslant n$. Since X can be embedded in Z, $\dim X \leqslant \dim Z$. The proof of lemma 2 is completed.

2. Main theorems. In order to state the theorems of this section, we need the following definitions:

Definitions. By an *n-compactification* of X we mean a compact space Y such that X is dense in Y and $\dim(Y \setminus X) = n$.

By the *deficiency* of X we mean the integer $\operatorname{def} X = \min\{n \mid$ for some Y, Y is an n-compactification of $X\}$. Clearly, $\operatorname{def} X$ is a topological invariant. This invariant was first defined by J. de Groot who also exhibited for each integer n $(n \geqslant -1)$ a space X with $\operatorname{def} X = n$. In [1] J. de Groot essentially proves

THEOREM 1. *X is semi-compact if and only if* $\operatorname{def} X \leqslant 0$.

We now prove a theorem which extends the theorem of A. Lelek.

THEOREM 2. *Let X be a space with the following three properties*:

(1) *each quasi-component Q of X is locally compact*;

(2) *for each quasi-component Q of X*, $\dim Q \leqslant n$;

(3) $\operatorname{def} X \leqslant n$.

Then $\dim X \leqslant n$.

Proof. Let Y be a k-compactification of X $(k = \operatorname{def} X)$ and Z be the subspace of $I \times Y$ defined in lemma 1. We need only show that $\dim Z_Q \leqslant n$ for each quasi-component Q of X. If $\bar{Q}$ is the closure of Q in Y, then $Q \subset \bar{Q} \subset \tilde{Q}$ since $\tilde{Q}$ is compact. Since Q is locally compact, Q is open relative to $\bar{Q}$. $\bar{Q} \setminus Q \subset \tilde{Q} \setminus Q \subset Y \setminus X$. Hence $\dim(\bar{Q} \setminus Q) \leqslant n$. By [2], Cor. 1, p. 32, $\dim \bar{Q} = \dim\big((\bar{Q} \setminus Q) \cup Q\big) \leqslant n$. Again by [2], Cor. 1, p. 32, $\dim \tilde{Q} = \dim\big((\tilde{Q} \setminus \bar{Q}) \cup \bar{Q}\big) \leqslant n$. Hence, by lemma 2, $\dim X \leqslant n$.

By combining theorems 1 and 2, we have

COROLLARY. *If X is a space which satisfies* (1) *and* (2) *of theorem* 2 *and furthermore is semi-compact, then* $\dim X \leqslant n$.

Finally, we extend a remark made in [4] concerning upper bounds for the difference between the dimension of a space and the maximum dimension of the quasi-components of X.

THEOREM 3. *Let X be a space with the property that* $\dim Q \leqslant n$ *for all quasi-components Q of X. Then* $\dim X \leqslant \operatorname{def} X + n + 1$.

Proof. Let Y be a k-compactification of X and Z be the subspace of $I \times Y$ defined in lemma 1. We need only prove that $\dim Z_Q \leqslant n + k + 1$

for all quasi-components Q of X. $\tilde{Q}\setminus Q \subset Y\setminus X$. Hence, $\dim(\tilde{Q}\setminus Q) \leqslant k$. By [2], proposition B), p. 28, $\dim\tilde{Q} \leqslant \dim(\tilde{Q}\setminus Q)+\dim Q+1 \leqslant n+k+1$. By lemma 2, $\dim X \leqslant n+k+1$. Let $k = \operatorname{def} X$.

When $\operatorname{def} X \leqslant 0$, we have the remark referred to above. By [2], Theorem V6, we have $\operatorname{def} X \leqslant \dim X$. If X is the n-dimensional space with zero dimensional quasi-components constructed by Mazurkiewicz [5], then theorem 3 implies $\operatorname{def} X \geqslant n-1$. Thus, we have examples of spaces with large deficiency.

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