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ON CHORDS OF CONVEX BODIES

BY

A. PLIŚ (CRACOW) AND A. TUROWICZ (TYNIEC)

In this note we give a theorem conjectured by T. Ważewski, which gives a limitation for the quotient of the lengths of segments of a chord passing through the centre of gravity of a convex body.

THEOREM. *Let c be the centre of gravity of an n-dimensional convex body (with an interior point) and a, b the end points of its chord passing through c. Then the following inequalities are satisfied:*

$$\frac{1}{n} \leqslant \frac{|a-c|}{|b-c|} \leqslant n. \tag{i}$$

Proof. Let A be an ($(n-1)$-dimensional) support plane (see Fig. 1) to the given convex body K at point a and C the plane passing through

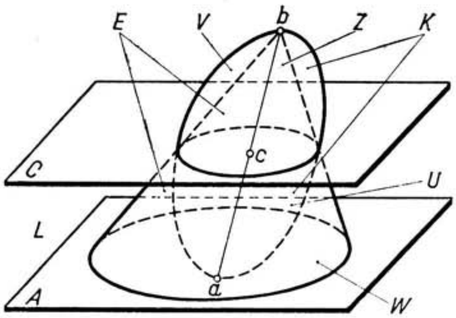

Fig. 1

c and parallel to A (we consider K in R^n). Let E be the cone consisting of all points lying between A and b on straight lines passing through b and any point of the set $K \frown C$. Denote by L the layer between A and C. Plane C cuts K into two parts: $U = K \frown L$ and $V = K - U$. Similarly, cone E is cut into two parts: $W = E \frown L$ and $Z = E - W$. Set K being convex, we have the inclusions

$$U \subset W \quad Z \subset V. \tag{ii}$$

Now we shall prove that the centre of gravity p (not marked on the figure) of E is situated between the planes A and C (on C if and only if $E = K$). Cone E is obtained from the set K by removing the set $V-Z$ lying outside layer L and adding the set $W-U$ lying inside layer L. Therefore the centre of gravity moves inside layer L, which can be shown by the following simple calculus. Consider an n-dimensional coordinate system with origin at c and x-axis perpendicular to C and directed towards A. Because c is the centre of gravity of $K = V \cup U$, we have $\int\limits_V x\,dv + \int\limits_U x\,dv = 0$, where the integrals are n-dimensional. In virtue of (ii) and the inequalities $x \geqslant 0$ on W, $x \leqslant 0$ on V we obtain $\int\limits_U x\,dv \leqslant \int\limits_W x\,dv$, $\int\limits_V x\,dv \leqslant \int\limits_Z x\,dv$. Therefore $\int\limits_E x\,dv \geqslant 0$ ($= 0$ if and only if $E = K$) and it follows that p belongs to layer L.

Let q be the point (not marked on the figure) of the chord lying on the plane passing through p and parallel to A. Obviously

$$\frac{|a-q|}{|q-b|} \leqslant \frac{|a-c|}{|c-b|}. \tag{iii}$$

It is well known that

$$\frac{|a-q|}{|q-b|} = \frac{1}{n}. \tag{iv}$$

In virtue of (iii) and (iv) we have the first inequality (i). The second one can be obtained by interchanging the roles of a and b.

Remark 1. The equalities in (i) are possible only for K being a cone (with the vertex at a or b respectively).

Remark 2. It is known [1] that the centre of gravity of an n-dimensional convex body is further from a support plane F than $1/(n+1)$

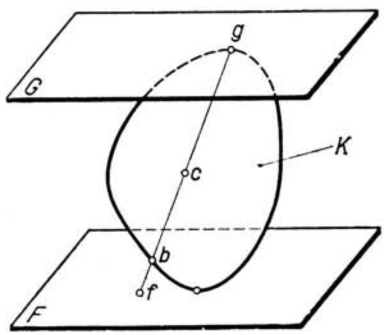

Fig. 2

[1] See T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Berlin 1934, p. 52-53.

times the width of the body in the direction perpendicular to the plane F, i.e. $1/(n+1)$ times the distance between F and the other support plane G parallel to F. This property can be deduced from our theorem. Indeed, let g be a point of the convex body K lying on G, f — the intersection point of F and the straight line N passing through the points g and c, and b the point on $N \frown K$ nearest to f (see Fig. 2). Obviously, $|f-c| \geqslant |b-c|$. Therefore by our theorem

$$\frac{|f-c|}{|c-g|} \geqslant \frac{|b-c|}{|c-g|} \geqslant \frac{1}{n}$$

and the property easily follows.

Similarly, our theorem can be deduced from this property.

MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

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