Friedman [1] has proved that if there exists a constant $K_0 > 0$ such that

$$\int_0^T \int_{E^m} \exp(-K_0|x|^2)|u(x,t)|\,dt\,dx < +\infty$$

and if the initial condition (5) is satisfied, then $u(x,t) \equiv 0$ (under the assumptions 1° and 2° of theorem I concerning the coefficients). Write

$$u^+(x,t) = \begin{cases} u(x,t), & \text{if} \quad u(x,t) \geqslant 0, \\ 0, & \text{if} \quad u(x,t) < 0. \end{cases}$$

There exists a supposition that the condition

$$\int_0^T \int_{E^m} \exp(-K_0|x|^2)u^+(x,t)\,dt\,dx < +\infty$$

is sufficient in order to make the solution $u(x,t)$ of equation (1), satisfying (5), vanish identically in $D^T$.

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# *EVALUATIONS OF SOLUTIONS OF A SECOND ORDER PARABOLIC EQUATION*

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Let us consider the equation

$$(1) \qquad \Delta u - \frac{\partial u}{\partial t} + c(x,t)u = 0, \quad \text{where} \quad \Delta = \sum_{i=1}^m \frac{\partial^2}{\partial x_i}, \; x = (x_1, \ldots, x_m).$$

The following theorem has been established by Krzyżański [4]:

THEOREM K. *Assume the coefficient $c(x,t)$ to be defined and continuous when $x$ varies in the m-dimensional Euclidean space $E^m$, $t > 0$, and to satisfy the Lipschitz's condition with respect to $x$. Suppose there exist constants $\alpha, \beta, A, B$; $\alpha > 0$, $A > 0$, $B > 0$, such that $\alpha^2|x|^2 + \beta \leqslant c(x,t) \leqslant A|x|^2 + B$ for $x \in E^m$, $t > 0$, where $|x| = \left(\sum_{i=1}^m x_i^2\right)^{1/2}$. If a solution $u(x,t)$ of equation (1) satisfies the condition $u(x,0) \geqslant N > 0$ for $x \in E^m$ and belongs to the so-called class $E_2$, then*

$$u(x,t) \geqslant M \exp(K|x|^2 \tan 2\alpha t) \quad \text{for} \quad x \in E^m, \; t \in \left(0, \frac{\pi}{4\alpha}\right),$$

*$M$, $K$, being positive constants.*

In the proof the author mentioned above has applied a fundamental solution, constructed in [7], which requires certain assumptions concerning a regularity of coefficients.

In this note we prove similar theorems for a more general equation of the form

$$(2) \qquad Fu \equiv \sum_{i,j=1}^m a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^m b_j(x,t)\frac{\partial u}{\partial x_j} + c(x,t)u - \frac{\partial u}{\partial t} = f(x,t)$$

by means of a method which does not use the fundamental solution. A quasi-linear equation will also be discussed.

The author is indebted to Professor M. Krzyżański for valuable remarks concerning this paper.

**1.** Let $D(h)$ be the topological product of $m$-dimensional Euclidean space $E^m$, of the variables $x_1, \ldots, x_m$, with the interval $(0, h)$, $0 < h \leqslant +\infty$. We introduce the following assumptions:

($H_1$) The coefficients $a_{ij}(x, t)$, $b_j(x, t)$, $c(x, t)$ and the free term $f(x, t)$ are defined in the domain $D(h)$;

($H_2$) There exist positive constants $A_0, \ldots, A_4$ such that

$$|a_{ij}(x, t)| \leqslant A_0, \quad |b_j(x, t)| \leqslant A_1|x|+A_2, \quad c(x, t) \leqslant A_3|x|^2+A_4 \text{ in } D(h),$$

where $|x| = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2}$.

($H_3$) $\sum_{i,j=1}^{m} a_{ij}(x, t)\xi_i\xi_j \geqslant 0$ for $(x, t)\in D(h)$.

By a solution of (2) is meant a function $u(x, t)$ which is continuous in the set $E^m \times \langle 0, h)$ and which has the derivative $\partial u/\partial t$ and continuous derivatives $\partial u/\partial x_i$, $\partial^2 u/\partial x_i \partial x_j$ in $D(h)$ satisfying (2).

THEOREM 1. *If the following conditions hold:*

1° *the assumptions* ($H_1$)-($H_3$) *are satisfied in* $D(h)$ ($h \leqslant +\infty$),

2° *there exist a positive function* $M(t)$ *continuous in the interval* $\langle 0, h)$ *and a positive constant* $K$ *such that a solution* $u(x, t)$ *of* (2) *satisfies the inequality*

$$u(x, t) \geqslant -M(t)\exp(K|x|^2) \quad (\textit{or } u(x, t) \leqslant M(t)\exp(K|x|^2))$$

*for* $(x, t)\in D(h)$ ($M(t)$ *may be unbounded in* $\langle 0, h)$),

3° $f(x, t) \leqslant 0$ ($f(x, t) \geqslant 0$ *respectively*) *in* $D(h)$,

4° $u(x, 0) \geqslant 0$ ($u(x, 0) \leqslant 0$ *respectively*), $x\in E^m$,

*then* $u(x, t) \geqslant 0$ ($u(x, t) \leqslant 0$ *respectively*) *in* $D(h)$.

The proof of this theorem is similar to the proof of theorem 1 of [2] (cf. also [1]), as for every domain $D(h_1)$, $h_1 < h$, there exists a constant $M_{h_1} > 0$ such that the inequality $u(x, t) \geqslant -M_{h_1}\exp(K|x|^2)$ is satisfied in $D(h_1)$.

THEOREM 2. *If the following conditions hold:*

1° ($H_1$)-($H_3$) *are satisfied in* $D(h)$,

2° *there exist a positive function* $M(t)$ *continuous in* $\langle 0, h)$ *and a positive constant* $K$ *such that a solution* $u(x, t)$ *of* (2) *satisfies the inequality* $u(x, t) \geqslant -M(t)\exp(K|x|^2)$ (*or* $u(x, t) \leqslant M(t)\exp(K|x|^2)$) *in* $D(h)$,

3° $f(x, t) \leqslant 0$ (*respectively* $f(x, t) \geqslant 0$),

4° $u(x, 0) \geqslant N$ (*respectively* $u(x, 0) \leqslant -N$); $N$ *being a positive constant,*

5° $c(x, t) \geqslant 0$ $(x, t)\in D(h)$,

*then* $u(x, t) \geqslant N$ (*respectively* $u(x, t) \leqslant -N$) *for* $(x, t)\in D(h)$.



Proof. Putting (for the case $f(x, t) \leqslant 0$) $\bar{u}(x, t) = u(x, t)-N$ we have $F\bar{u} = -c(x, t)N+f(x, t) \leqslant 0$ and $\bar{u}(x, 0) \geqslant 0$. Therefore, for the function $\bar{u}(x, t)$, all the assumptions of theorem 1 are fulfilled, whence $u(x, t)-N \geqslant 0$.

We shall prove

THEOREM 3. *If the assumptions* 1°-4° *of theorem* 2 *are fulfilled and if there is a constant* $\gamma$ *such that* $c(x, t) \geqslant \gamma$ *in* $D(h)$, *then*

$$u(x, t) \geqslant N\exp(\gamma t) \quad (u(x, t) \leqslant -N\exp(\gamma t) \textit{ respectively})$$

*for* $(x, t)\in D(h)$.

Proof. The substitution

$$u(x, t) = v(x, t)\exp(\gamma t)$$

transforms (2) into

$$\sum_{i,j=1}^{m} a_{ij}(x, t)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{j=1}^{m} b_j(x, t)\frac{\partial v}{\partial x_j} + \bar{c}(x, t)v - \frac{\partial v}{\partial t} = f(x, t)\exp(-\gamma t)$$

with $\bar{c}(x, t) = [c(x, t)-\gamma]\exp(-\gamma t) \geqslant 0$. Furthermore, the function $v(x, t)$ satisfies the inequality $v(x, 0) \geqslant N$ in the case $u(x, 0) \geqslant N$. Now, for $v(x, t)$ all the assumptions of theorem 2 are fulfilled. Hence $v(x, t) \geqslant N$ or $u(x, t) \geqslant N\cdot\exp(\gamma t)$. In the case $u(x, 0) \leqslant N$ the proof is similar.

THEOREM 4 [1]. *If the following conditions hold:*

1° *there exist constants* $\alpha$ *and* $\beta$, $\alpha > 0$, *such that the inequality* $c(x, t) \geqslant \alpha^2|x|^2+\beta$ *holds in the domain* $D(h_0)$, *where* $h_0 < \pi/4\alpha L$, $L$ *being a positive constant (see assumption* 3°),

2° *the hypotheses* ($H_1$)-($H_3$) *are satisfied in* $D(h_0)$, $\alpha^2 \leqslant A_3$, $\beta \leqslant A_4$,

3° $\sum_{i,j=1}^{m} a_{ij}(x, t)x_i x_j \geqslant L^2|x|^2$, $\sum_{j=1}^{m} b_j(x, t)x_j \geqslant 0$ [2] *for* $(x, t)\in D(h_0)$,

4° $f(x, t) \leqslant 0$ (*or* $f(x, t) \geqslant 0$),

5° $u(x, t) \geqslant -M(t)\cdot\exp(K|x|^2)$ ($u(x, t) \leqslant M(t)\exp(K|x|^2)$ *respectively*) *in the domain* $D(h_0)$, $M(t)$ *being a positive function continuous in* $\langle 0, h_0)$, $K > 0$,

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[1] I have learned that a similar theorem has been proved by Krzyżański [6]. His theorem requires weaker assumptions than theorem 4 but constitutes a less precise estimate.

[2] This inequality may be replaced by the following one: $\sum_{j=1}^{m} b_j(x, t)x_j \geqslant -\sum_{i=1}^{m} a_{ii}(x, t)$, which is a slightly weaker condition.

6° $u(x, 0) \geqslant N > 0$ ($u(x, 0) \leqslant -N < 0$ *respectively*), *then the inequality*

$$(3)\qquad u(x, t) \geqslant N \cdot \exp\left(\frac{a}{2L}|x|^2 \cdot \tan 2aLt + \beta t\right)$$

($u(x, t) \leqslant -N \cdot \exp\left(\frac{a}{2L}|x|^2 \cdot \tan 2aLt + \beta t\right)$ *respectively*) *is satisfied in the domain* $D(h_0)$.

Proof. Put

$$(4)\qquad H = \exp\left(\frac{a}{2L}|x|^2 \tan 2aLt + \beta t\right).$$

The substitution

$$(5)\qquad u(x, t) = v(x, t) \cdot H, \quad (x, t) \in D(h_0)$$

transforms (2) into the equation

$$(6)\qquad \sum_{i,j=1}^{m} a_{ij}(x, t)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{j=1}^{m} \bar{b}_j(x, t)\frac{\partial v}{\partial x_j} + \bar{c}(x, t) u - \frac{\partial v}{\partial t} = \frac{1}{H} f(x, t),$$

where

$$\bar{c}(x, t) = \frac{FH}{H}, \quad \bar{b}_j(x, t) = \frac{1}{H}\left(2\sum_{i=1}^{m} a_{ij}(x, t)\frac{\partial H}{\partial x_i} + b_j(x, t) H\right).$$

We shall prove that for $v(x, t)$ the assumptions of theorem 2 are satisfied. First we shall show that $\bar{c}(x, t) \geqslant 0$ in $D(h_0)$. Indeed,

$$\bar{c}(x, t) = \frac{FH}{H} = \frac{a^2}{L^2}\tan^2 2aLt \sum_{i,j=1}^{m} a_{ij} x_i x_j + \frac{a}{L}\tan 2aLt \sum_{i=1}^{m} a_{ii} + \frac{a}{L}\tan 2aLt \sum_{j=1}^{m} b_j x_j + c(x, t) - a^2|x|^2 \cos^{-2} 2aLt - \beta.$$

By the assumption that

$$\sum_{i,j=1}^{m} a_{ij}(x, t)\xi_i \xi_j \geqslant 0$$

we have $a_{ii} \geqslant 0$ $(i = 1, \ldots, m)$. Taking, moreover, into account 1° and 3°, we derive $\bar{c}(x, t) \geqslant 0$. From our assumptions it also follows that there exist positive constants $\bar{A}_3$, $\bar{A}_4$ such that $\bar{c}(x, t) \leqslant \bar{A}_3|x|^2 + \bar{A}_4$ in the zone $D(h_0)$, $h_0 < \pi/4aL$. It can easily be shown that $|\bar{b}_j(x, t)| \leqslant \bar{A}_1|x| + \bar{A}_2$ $(j = 1, \ldots, m)$ in $D(h_0)$, $\bar{A}_1$, $\bar{A}_2$ being certain positive constants. Notice that for the function $v(x, t)$ the remaining assumptions of theorem 2 are fulfilled too. In particular, we have $v(x, 0) \geqslant N$. Consequently we get $v(x, t) \geqslant N$ for $(x, t) \in D(h_0)$. Hence and from 5° we obtain the first part of thesis (3).



Remark. We will give an example which shows that if the assumption $\sum_{j=1}^{m} b_j(x, t) x_j \geqslant 0$ is not satisfied, then the assertion of theorem 4 is false. Namely, it is easy to verify that the function $u(x, t) = \exp(x^2 t)$ satisfies the equation

$$u''_{xx} - 2xtu'_x + (x^2 - 2t)u - u'_t = 0,$$

$u(x, 0) = 1$, and does not satisfy (3) in the whole domain $D(h_0)$ if $h_0$ is sufficiently near $\pi/4$. We have $\sum_{j=1}^{m} b_j(x, t) x_j = -2x^2 t < 0$ when $x \neq 0$, $t > 0$. The remaining assumptions of theorem 4 are fulfilled.

**2.** Let us now consider the quasilinear equation

$$(7)\qquad \sum_{i,j=1}^{m} a_{ij}\left(x, t, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_m}\right)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{m} b_j\left(x, t, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_m}\right)\frac{\partial u}{\partial x_j} + c\left(x, t, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_m}\right)u - \frac{\partial u}{\partial t} = f\left(x, t, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_m}\right),$$

where the coefficients and the function $f$ contain the unknown function $u(x, t)$ and its partial derivatives $\partial u/\partial x_k$ $(k = 1, \ldots, m)$ of the first order. Similarly as in section 1 the following conditions are introduced:

($\bar{\mathrm{H}}_1$) The coefficients $a_{ij}(x, t, u, z_1, \ldots, z_m)$, $b_j(x, t, u, z_1, \ldots, z_m)$, $c(x, t, u, z_1, \ldots, z_m)$, and the function $f(x, t, u, z_1, \ldots, z_m)$ to be defined in the domain $\Pi(h)$: $(x, t) \in D(h)$ (see sec. 1), $u, z_1, \ldots, z_m$ arbitrary,

($\bar{\mathrm{H}}_2$) There are positive constants $A_0, \ldots, A_4$ such that

$$|a_{ij}(x, t, u, z_1, \ldots, z_m)| \leqslant A_0, \quad |b_j(x, t, u, z_1, \ldots, z_m)| \leqslant A_1|x| + A_2,$$
$$c(x, t, u, z_1, \ldots, z_m) \leqslant A_3|x|^2 + A_4 \quad \text{in } \Pi(h),$$

($\bar{\mathrm{H}}_3$) $\sum_{i,j=1}^{m} a_{ij}(x, t, u, z_1, \ldots, z_m)\xi_i \xi_j \geqslant 0$ in $\Pi(h)$.

Similar changes in formulating the assumptions of theorems 1-4 of the previous section permit to conclude that those theorems hold if

$u(x, t)$ is the solution of equation (7). Indeed, let $u(x, t)$ be an arbitrary solution of equation (7) defined on the domain $D(h)$. If we put

$$a_{ij}^{(1)}(x, t) = a_{ij}\left(x, t, u(x, t), \ldots, \frac{\partial u(x, t)}{\partial x_k}, \ldots\right),$$

$$b_j^{(1)}(x, t) = b_j\left(x, t, u(x, t), \ldots, \frac{\partial u(x, t)}{\partial x_k}, \ldots\right),$$

$$c^{(1)}(x, t) = c\left(x, t, u(x, t), \ldots, \frac{\partial u(x, t)}{\partial x_k}, \ldots\right),$$

$$f^{(1)}(x, t) = f\left(x, t, u(x, t), \ldots, \frac{\partial u(x, t)}{\partial x_k}, \ldots\right),$$

then $u(x, t)$ may be considered as a solution of a linear equation of the form (2). For instance the following two theorems hold true:

THEOREM 1'. *If $u(x, t)$ is a solution of equation* (7) *satisfying the assumptions* 2° *and* 4° *of theorem* 1 (sec. 1) *and if the conditions* $(\overline{\mathrm{H}}_1)$-$(\overline{\mathrm{H}}_3)$ *are satisfied in $\Pi(h)$ as well as $f(x, t, u, z_1, \ldots, z_m) \leqslant 0$ ($f(x, t, u, z_1, \ldots, z_m) \geqslant 0$ respectively), then $u(x, t) \geqslant 0$ ($u(x, t) \leqslant 0$ respectively) in $D(h)$.*

THEOREM 4'. *Suppose there exist constants $a > 0$ and $\beta$ such that $c(x, t, u, z_1, \ldots, z_m) \geqslant a^2|x|^2+\beta$ in $\Pi(h_0)$, $h_0 < \pi/4aL$, $L > 0$. Let the conditions $(\overline{\mathrm{H}}_1)$-$(\overline{\mathrm{H}}_3)$ be satisfied and let the assumptions 5°, 6° of theorem 4 hold true in $\Pi(h_0)$. Suppose, furthermore, that*

$$\sum_{i,j=1}^{m} a_{ij}(x, t, u, z_1, \ldots, z_m)x_ix_j \geqslant L^2|x|^2, \qquad \sum_{j=1}^{m} b_j(x, t, u, z_1, \ldots, z_m)x_j \geqslant 0,$$

*and*

$$f(x, t, u, z_1, \ldots, z_m) \leqslant 0 \qquad (f(x, t, u, z_1, \ldots, z_m) \geqslant 0 \text{ respectively})$$

*in $\Pi(h_0)$. Then the solution $u(x, t)$ of* (7) *satisfies the inequalities* (3) *in $D(h_0)$.*

Likewise we deduce that the theorem given by Krzyżański in section 4 of [4] may be formulated for the solution $u(x, t)$ of equation (7) as follows:

THEOREM 5. *If the following conditions hold:*

1° *the conditions $(\overline{\mathrm{H}}_1)$-$(\overline{\mathrm{H}}_3)$ are satisfied and $f(x, t, u, z_1, \ldots, z_m) \equiv 0$ in $\Pi(h)$, $h \leqslant +\infty$,*

2° *there are positive constants $N$, $K$, and a positive function $M(t)$ continuous in $\langle 0, h)$ such that $|u(x, t)| \leqslant M(t)\cdot\exp(K|x|^2)$ in $D(h)$ and $u(x, 0) \geqslant N$ for $x \in E^m$,*



3° *there exist $a > 0$ and $\beta$ such that $c(x, t, u, z_1, \ldots, z_m) \leqslant -a^2|x|^2+\beta$ for $(x, t) \in D(h)$,*

*then the inequality*

$$|u(x, t)| \leqslant N\cdot\exp(-\lambda|x|^2\cdot\tan h\mu t+\nu t)$$

*holds for $(x, t) \in D(h)$, where the number $\mu$ is arbitrarily chosen and $\lambda$ ($> 0$), $\nu$, depend on the coefficients $a_{ij}$, the numbers $A_1$, $A_2$, appearing in $(\overline{\mathrm{H}}_2)$, and on the constants $a$, $\beta$.*

Theorems similar to theorems 1, 2 and 3 may also be obtained for the second and third Fourier's problems in the domains considered in [3].

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