take on the respective forms,

$$F = FI2K, \quad FF = F1FI.$$

The functional equation of commutativity thus has order 2, while the superficially more complicated functional equation of associativity has the lower order 1.

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A SUFFICIENT CONDITION FOR INDEPENDENCE

BY

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The present paper originated in an attempt to decide whether the well known notion of independence of continuous functions comes under the general scheme of such notions given by E. Marczewski [1]. Our theorem gives a positive answer to this and to a class of similar questions.

By an *algebra* we mean what is also called a general algebraic system, i. e. a set A together with some A-valued functions of finitely many variables defined over A (also called *operations*). The notion of independence in an algebra is taken from the paper of E. Marczewski and it is assumed that the reader is acquainted with it.

THEOREM. *Let A be an arbitrary set with at least two elements, $\overline{\overline{A}} \geqslant 2$. Let $\boldsymbol{J} \subset 2^A$ be a hereditary family of finite subsets of A and let $S = \{x : \{x\} \notin \boldsymbol{J}\}$. Then the condition $\overline{\overline{S}} \geqslant \overline{\overline{\boldsymbol{J}}}$ implies the existence of an algebra with the set of elements A in which $\boldsymbol{J}$ is the class of exactly all finite independent subsets.*

If we wish $\boldsymbol{J}$ to be the family of all finite independent subsets of an algebra, then clearly every element of S has to form a dependent set, i. e. it has to be self-dependent. Thus our theorem says that if $\boldsymbol{J}$ allows for sufficiently many self-dependent elements, then it can be the family of all finite independent subsets in a suitable algebra over A.

If $\overline{\overline{S}} \geqslant \overline{\overline{\boldsymbol{J}}}$ does not hold, then there may be no algebra over A in which $\boldsymbol{J}$ is the set of exactly all finite independent subsets. There is no such algebra if $A = \{a, b, c\}$ and $\boldsymbol{J} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ ($\emptyset$ denotes the empty set). Assuming the contrary, we have from the self-dependence of c that there is an operation $f(x)$ such that $f(x) \neq x$ holds for some $x \in A$. Then it follows from the independence of a that $f(a) \neq a$. Since

[1] E. Marczewski, *A general scheme of the notions of independence in mathematics*, Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques, 6 (1958), p. 731-736.

a, b are independent, we have also $f(a) \neq b$, whence $f(a) = c$. By symmetry, $f(b) = c$. Hence $f(a) = f(b)$, contradicting the independence of a, b.

Proof of the theorem. Let A and $\boldsymbol{J} \subset 2^A$ satisfy the above assumptions. If $\boldsymbol{J} = \{\emptyset\}$, then our task is to construct an algebra on A which has no independent elements at all. This can be done as follows. As there are at least two elements $0, 1 \in A$, we can define an operation $f(x)$ on A by putting $f(0) = 1$, $f(x) = x$ for $x \neq 0$, and another operation $g(x)$ such that $g(1) = 0$ and $g(x) = x$ for $x \neq 1$. Then we have, for each x, either $f(x) = x$ or $g(x) = x$, or both. Hence x cannot be independent in the algebra $(A; f, g)$ because this would imply that one of these equalities has to hold identically.

Suppose now that some non-empty set belongs to $\boldsymbol{J}$. Let $m \geqslant 1$ be the greatest integer such that there is a set with m elements in $\boldsymbol{J}$, or let m be infinity. S being non-empty, we can pick out one element $0 \in S$. Since $S - \{0\}$ contains at least as many elements as $\boldsymbol{J} - \{\emptyset\}$, there is a one-to-one mapping q of $\boldsymbol{J} - \{\emptyset\}$ into $S - \{0\}$. Using this mapping we define operations $f_1(x_1), f_2(x_1, x_2), \ldots, f_n(x_1, \ldots, x_n), \ldots$ for all integers n which do not exceed m as follows:

$$f_n(a_1, \ldots, a_n) = \begin{cases} q\{a_1, \ldots, a_n\} & \text{if all } a_i \text{ are different and } \{a_1, \ldots, a_n\} \in \boldsymbol{J}, \\ 0 & \text{in all other cases.} \end{cases}$$

We note that none of these operations assumes identically the value 0. An operation assuming the value 0 on any number of arguments will also be needed and we shall denote it by f_∞. It is easily seen that we have always $f_n(a_1, \ldots, a_n) \in S$ if $n \leqslant m$.

For integers k satisfying $m < k \leqslant \overline{\overline{A - S}}$ (if such exist) we define operations $f_k(x_1, \ldots, x_k)$ as follows:

$$f_k(a_1, \ldots, a_k) = \begin{cases} a_k & \text{if } a_i \notin S \ (i = 1, \ldots, k), \text{ and all } a_i \text{ are different,} \\ 0 & \text{in all other cases.} \end{cases}$$

Since there are at least k elements in $A - S$, we have $f_k \neq f_\infty$. It is also clear that $f_k(x_1, \ldots, x_k)$ does not assume identically the value x_k.

It is now our assertion that the algebra we need is

$$\mathscr{A} = (A; f_\infty, f_1, f_2, \ldots);$$

that is to say, $X \subset A$ is a set of independent elements in $\mathscr{A}$ if and only if $X \in \boldsymbol{J}$.

First we note that if $\{a_1, \ldots, a_r\} \notin \boldsymbol{J}$, then $a_1, \ldots, a_r$ are dependent in $\mathscr{A}$. This is trivially so if $r \leqslant m$, for then $f_r(a_1, \ldots, a_r) = 0 = f_\infty(a_1, \ldots, a_r)$, whereas $f_r \neq f_\infty$. If $r > m$ and $r \leqslant \overline{\overline{A - S}}$, then we have either

$f_r(a_1, \ldots, a_r) = f_\infty(a_1, \ldots, a_r)$ as before, or $f_r(a_1, \ldots, a_r) = a_r$, and hence the dependence of $a_1, \ldots, a_r$. Finally, if $r > \overline{\overline{A - S}}$, then one of the elements $a_1, \ldots, a_r$ must belong to S, say $a_i \in S$, and we have $f_1(a_i) = 0 = f_\infty(a_i)$. This proves that a_i is dependent in $\mathscr{A}$ and consequently $a_1, \ldots, a_r$ are dependent.

For the remaining part of our proof we need to determine, for every $n \leqslant m$, the class $\boldsymbol{A}^{(n)}$ of all operations of n variables in $\mathscr{A}$. We call the operations $f_\infty, f_1(x_1), f_2(x_1, x_2), \ldots$ fundamental. $\boldsymbol{A}^{(n)}$ is, by definition, the smallest class such that

(*) all identity operations $e_i(x_1, \ldots, x_n) \equiv x_i$, $i = 1, \ldots, n$, belong to $\boldsymbol{A}^{(n)}$,

(**) if f_r is a fundamental operation, $r < \infty$, and $h_j \in \boldsymbol{A}^{(n)}$, $j = 1, \ldots, r$, then

$$f_r\big(h_1(x_1, \ldots, x_n), \ldots, h_r(x_1, \ldots, x_n)\big) \in \boldsymbol{A}^{(n)},$$

(***) $f_\infty \in \boldsymbol{A}^{(n)}$.

Now let $\boldsymbol{C}_n$ be the class of those operations $h(x_1, \ldots, x_n)$ which either coincide with one of the e_i (cf. (*)) or else satisfy an identity of the form

$$(1) \qquad h(x_1, \ldots, x_n) \equiv f_l(x_{i_1}, \ldots, x_{i_l}) \quad \text{for all} \quad x_1, \ldots, x_n \in A,$$

where $l \leqslant \infty$ and $i_1, \ldots, i_l \in \{1, \ldots, n\}$ are fixed. We shall prove that if $n \leqslant m$, then $\boldsymbol{C}_n = \boldsymbol{A}^{(n)}$.

First it is clear that both (*) and (***) hold for $\boldsymbol{C}_n$. So we have to verify still that $\boldsymbol{C}_n$ is closed with respect to taking combinations such as required by (**). To do this we assume that f_r $(r < \infty)$, is a fundamental operation and $h_1, \ldots, h_r \in \boldsymbol{C}_n$. Then it is obvious that if all h_i are identity operations, then $f_r(h_1, \ldots, h_r)$ is of the form (1) and thus belongs to $\boldsymbol{C}_n$. If at least one of the h_i is not an identity operation, thus being of the form (1), then all values attained by this operation lie in S. This follows from $f_l(x_{i_1}, \ldots, x_{i_l}) \equiv 0 \in S$ if $l > n$ (for then at least two among the variables $x_{i_1}, \ldots, x_{i_l}$ must be identical), and from the fact that $f_l(x_{i_1}, \ldots, x_{i_l}) \in S$ for all values of $x_{i_1}, \ldots, x_{i_l}$ if $l \leqslant n \leqslant m$. Therefore we have in this case $f_r(h_1, \ldots, h_r) \equiv 0$. Hence this operation belongs to $\boldsymbol{C}_n$.

To complete the proof of the theorem we have to show that $\{a_1, \ldots, a_n\} \in \boldsymbol{J}$, $n \leqslant m$, implies the independence of $a_1, \ldots, a_n$. Suppose that $\{a_1, \ldots, a_n\} \in \boldsymbol{J}$ and let there be two operations $g, h \in \boldsymbol{A}^{(n)}$ for which

$$(2) \qquad g(a_1, \ldots, a_n) = h(a_1, \ldots, a_n).$$

We have to show that $g = h$. As n does not exceed m, we have here $\boldsymbol{A}^{(n)} = \boldsymbol{C}_n$, whence g is either an identity operation, or it is of the form (1), and similarly h. Now it is true that if one of them is an identity operation, then so is the other, for it should be noted that $e_i(a_1, \ldots, a_n) = a_i \notin S$ while the values attained by operations of the form (1) are always in S (as shown above). Certainly $g = e_i$ and $h = e_j$ implies $i = j$ and then we have $g = h$. In all other cases we may assume that h is given by (1) and similarly

$$g(x_1, \ldots, x_n) \equiv f_r(x_{j_1}, \ldots, x_{j_r}), \tag{3}$$

where $r \leqslant \infty$, $j_1, \ldots, j_r \epsilon \{1, \ldots, n\}$.

It is easily seen that if an operation h of the form (1) satisfies $h(a_1, \ldots, a_n) = 0$, where $\{a_1, \ldots, a_n\} \epsilon \boldsymbol{J}$, then either $l = \infty$ or some number occurs at least twice in the sequence $i_1, \ldots, i_l$. In both cases we have identically $h(x_1, \ldots, x_n) \equiv 0$. The same being true for g, the appearance of 0 in (2) implies $g(x_1, \ldots, x_n) \equiv 0 \equiv h(x_1, \ldots, x_n)$.

If $g(a_1, \ldots, a_n) = h(a_1, \ldots, a_n) \neq 0$, then we must have $l, r \leqslant n$ (cf. (1), (3)) and, by $n \leqslant m$,

$$\begin{aligned} f_l(a_{i_1}, \ldots, a_{i_l}) &= q\{a_{i_1}, \ldots, a_{i_l}\}, \\ f_r(a_{j_1}, \ldots, a_{j_r}) &= q\{a_{j_1}, \ldots, a_{j_r}\}. \end{aligned} \tag{4}$$

It follows now from the one-to-one property of the mapping q, by (1), (2), (3) and (4) that $\{a_{i_1}, \ldots, a_{i_l}\} = \{a_{j_1}, \ldots, a_{j_r}\}$. Hence $l = r$, $\{i_1, \ldots, i_l\} = \{j_1, \ldots, j_r\}$ and, again by (1) and (3), $g = h$. This completes our proof.

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REMARKS ON THE CARTESIAN PRODUCT OF TWO GRAPHS

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1. In paper [4] H. S. Shapiro introduced a notion of the *Cartesian product* $G_1 \times G_2$ of two graphs. F. Harary in his paper [2] (see also [3]) introduced a notion of the *composition* $G_1 * G_2$ (we write $G_1 * G_2$ instead of Harary's notation $G_1[G_2]$, according to the associativity of this operation) of two graphs. These notions for connected graphs are special cases of a more general notion of the Cartesian product of two graphs with metrics. In the present note we shall study this product under some natural assumptions concerning these *metrics*, namely those of [1] (p. 630). We shall prove that under these assumptions our product coincides with $G_1 \times G_2$ or $G_1 * G_2$.

2. Definitions. A pair $\langle N, \varrho\rangle$, where N is a finite or infinite set, is said to be an *NS-space* if $\varrho(x, y)$ is a function defined on the whole N whose values are non-negative integers such that

1° $\varrho(x, y) = 0$ if and only if $x = y$,

2° $\varrho(x, y) = \varrho(y, x)$,

3° $\varrho(x, y) + \varrho(y, z) \geqslant \varrho(x, z)$,

4° If $\varrho(x, y) = n$ $(n \geqslant 1)$, then there exists an element $z \epsilon N$ such that $\varrho(x, z) = 1$ and $\varrho(z, y) = n-1$.

The *Cartesian product* of two *NS*-spaces $\langle N_1, \varrho_1\rangle$ and $\langle N_2, \varrho_2\rangle$ we define as an *NS*-space $\langle N_1 \times N_2, \varrho\rangle$, where $N_1 \times N_2$ is the set of ordered pairs (x, y), $x \epsilon N_1$, $y \epsilon N_2$, with the metric ϱ defined by

$$\varrho[(x_1, y_1), (x_2, y_2)] = f[\varrho_1(x_1, x_2), \varrho_2(y_1, y_2)] = f(k, m),$$

$k = \varrho_1(x_1, x_2)$, $m = \varrho_2(y_1, y_2)$, $x_1, x_2 \epsilon N_1$, $y_1, y_2 \epsilon N_2$, where f is a function whose values are non-negative integers and satisfies the following conditions (see [1], p. 630):

(1) $f(k, m) = f(m, k)$ for all non-negative integers m, k,