COROLLARY. *Let f be D_*-integrable on $[a,b]$. In order that for every function φ which is ACG_* on an interval $[c,d]$ and such that $\varphi[[c,d]]\subset\subset[a,b]$, the function $f(\varphi)\varphi'$ be D_*-integrable on $[c,d]$ and (1) hold, it is necessary and sufficient that an indefinite D_*-integral of f on $[a,b]$ be the function LG_*.*

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ON A RECURRENCE RELATION

BY

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In the present paper we shall consider the recurrence relation

$$x_{n+1}+x_n=b_n, \tag{1}$$

in which the sequence b_n is given and x_n is to be determined. Of course, the sequence x_n can be found in infinitely many ways. We may choose arbitrarily the term x_0 and then the whole sequence x_n will be uniquely determined by relation (1). However, we shall prove that under suitable assumptions there exists only one sequence x_n fulfilling relation (1) and an additional condition. In what follows all occurring sequences are supposed to be real.

For an arbitrary sequence a_n we denote (as usual) by Δa_n the difference

$$\Delta a_n \stackrel{\text{df}}{=} a_{n+1}-a_n.$$

Further, we define the successive iterates of the operator Δ by the relations

$$\Delta^0 a_n \stackrel{\text{df}}{=} a_n, \qquad \Delta^{\nu+1} a_n \stackrel{\text{df}}{=} \Delta\Delta^{\nu} a_n, \qquad \nu=0,1,2,\ldots$$

Of course, the operator Δ^1 coincides with the operator Δ.

The purpose of the present note is to prove the following

THEOREM. *If (for a certain $r\geqslant 1$) the terms $\Delta^{r+1} b_n$ have a constant sign, and for a certain positive integer $p\leqslant r$*

$$\lim_{n\to\infty}\Delta^p b_n=0, \tag{2}$$

then there exists exactly one sequence x_n such that the terms $\Delta^r x_n$ have a constant sign, and relation (1) holds. This sequence is given by the formula

$$x_n=\sum_{\nu=0}^{p-1}\frac{(-1)^\nu}{2^{\nu+1}}\Delta^\nu b_n+\frac{(-1)^p}{2^p}\sum_{\nu=0}^{\infty}(-1)^\nu\Delta^p b_{n+\nu}. \tag{3}$$

The proof of the above theorem will be based on some lemmas.

LEMMA I. *For an arbitrary sequence a_n we have*

$$\sum_{\nu=0}^{p}\frac{(-1)^\nu}{2^{\nu+1}}\{\Delta^\nu a_{n+1}+\Delta^\nu a_n\} = a_n-\frac{(-1)^{p+1}}{2^{p+1}}\Delta^{p+1}a_n,\quad p=0,1,2,\dots \tag{4}$$

Proof. The proof will be by induction. For $p=0$ formula (4) is evident. Assuming its validity for $p-1\geqslant 0$ we have for p

$$\sum_{\nu=0}^{p}\frac{(-1)^\nu}{2^{\nu+1}}\{\Delta^\nu a_{n+1}+\Delta^\nu a_n\} = a_n-\frac{(-1)^p}{2^p}\Delta^p a_n+\frac{(-1)^p}{2^{p+1}}\{\Delta^p_{a_{n+1}}+\Delta^p_{a_n}\}$$

$$= a_n-\frac{(-1)^{p+1}}{2^{p+1}}\{\Delta^p a_{n+1}-\Delta^p a_n\} = a_n-\frac{(-1)^{p+1}}{2^{p+1}}\Delta^{p+1}a_n,$$

which completes the proof of the lemma.

The two following lemmas guarantee the uniqueness of the sequence fulfilling relation (1) and some additional conditions.

LEMMA II. *If a sequence x_n satisfies relation* (1) *and* (*for a fixed integer $p\geqslant 1$*) *fulfills the condition*

$$\lim_{n\to\infty}\Delta^p x_n = 0, \tag{5}$$

then it must have form (3).

Proof. Applying the operation Δ^p to both sides of relation (1) we obtain

$$\Delta^p x_{n+1}+\Delta^p x_n = \Delta^p b_n.$$

Putting $y_n \overset{\mathrm{df}}{=} \Delta^p x_n$ and $c_n \overset{\mathrm{df}}{=} \Delta^p b_n$, we have evidently $\Delta^p x_{n+1} = y_{n+1}$, and thus the above relation may be written as

$$y_{n+1}+y_n = c_n.$$

According to (5) $\lim\limits_{n\to\infty} y_n = 0$ whence, on account of the relation

$$y_n = \sum_{\nu=0}^{k}(-1)^\nu c_{n+\nu}+(-1)^{k+1}y_{n+k+1}$$

(easily obtainable by induction) we have

$$y_n = \sum_{\nu=0}^{\infty}(-1)^\nu c_{n+\nu},$$

i. e.

$$\Delta^p x_n = \sum_{\nu=0}^{\infty}(-1)^\nu\Delta^p b_{n+\nu}. \tag{6}$$

Further, on account of (1) and of the definition of the operators Δ^i we have for $0\leqslant i<p$

$$\Delta^i x_{n+1}+\Delta^i x_n = \Delta^i b_n,\qquad \Delta^i x_{n+1}-\Delta^i x_n = \Delta^{i+1}x_n,$$

whence

$$\Delta^i x_n = \tfrac{1}{2}\{\Delta^i b_n-\Delta^{i+1}x_n\}. \tag{7}$$

Using successively formula (7) for $i=0,1,\dots,p-1$, and taking into account relation (6), we obtain formula (3), which was to be proved.

LEMMA III. *If for a fixed integer $p\geqslant 1$ relation* (2) *holds, then a sequence x_n satisfying relation* (1) *and such that the terms $\Delta^p x_n$ have a constant sign, must have form* (3).

Proof. Replacing n by $n+k$ in relation (1) and then applying to both sides of (1) the operation Δ^p we obtain

$$\Delta^p x_{n+k+1}+\Delta^p x_{n+k} = \Delta^p b_{n+k},$$

whence we have by (2)

$$\lim_{k\to\infty}\{\Delta^p x_{n+k+1}+\Delta^p x_{n+k}\} = 0. \tag{8}$$

Since the terms $\Delta^p x_{n+k}$ have a constant sign, it follows from (8) that

$$\lim_{n\to\infty}\Delta^p x_n = 0,$$

whence on account of lemma II we obtain formula (3). This completes the proof.

Now we proceed to prove the theorem formulated in the beginning of this paper.

Proof of the theorem. Since $\Delta^{r+1}b_n$ have a constant sign and $r\geqslant p\geqslant 1$, the terms $\Delta^{p+1}b_n$ have a constant sign for n sufficiently large, and consequently, the sequence $\Delta^p b_n$ is monotonic for n sufficiently large. Thus the series

$$\sum_{\nu=0}^{\infty}(-1)^\nu\Delta^p b_{n+\nu}$$

converges, since it is an alternating series.

Consequently formula (3) actually defines a sequence x_n. We have

$$x_{n+1} = \sum_{\nu=0}^{p-1}\frac{(-1)^\nu}{2^{\nu+1}}\Delta^\nu b_{n+1}+\frac{(-1)^p}{2^p}\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\Delta^p b_{n+\nu}, \tag{9}$$

whence

$$\Delta x_n = \sum_{\nu=0}^{p-1}\frac{(-1)^\nu}{2^{\nu+1}}\Delta^{\nu+1}b_n + \frac{(-1)^p}{2^p}\Big\{\sum_{\nu=1}^{\infty}[(-1)^{\nu-1}-(-1)^\nu]\Delta^p b_{n+\nu}\Big\} - \frac{(-1)^p}{2^p}\Delta^p b_n$$

$$= \sum_{\nu=0}^{p-1}\frac{(-1)^\nu}{2^{\nu+1}}\Delta^{\nu+1}b_n + \frac{(-1)^{p-1}}{2^{p-1}}\sum_{\nu=0}^{\infty}(-1)^\nu\Delta^p b_{n+\nu} - \frac{(-1)^{p-1}}{2^p}\Delta^p b_n$$

$$= \sum_{\nu=0}^{p-2}\frac{(-1)^\nu}{2^{\nu+1}}\Delta^{\nu+1}b_n + \frac{(-1)^{p-1}}{2^{p-1}}\sum_{\nu=0}^{\infty}(-1)^\nu\Delta^p b_{n+\nu}.$$

Repeating this procedure applied successively for Δx_n, $\Delta^2 x_n$, etc. p-times, we obtain finally

$$\Delta^p x_n = \sum_{\nu=0}^{\infty}(-1)^\nu\Delta^p b_{n+\nu}.$$

Applying the operation Δ^{r-p} to both sides of the above equality we get

$$\Delta^r x_n = \sum_{\nu=0}^{\infty}(-1)^\nu\Delta^r b_{n+\nu}.$$

The series occurring on the right-hand side of the above relation may be also written in the form

$$-\sum_{\nu=0}^{\infty}\{\Delta^r b_{n+2\nu+1}-\Delta^r b_{n+2\nu}\} = -\sum_{\nu=0}^{\infty}\Delta^{r+1}b_{n+2\nu}.$$

Since the terms $\Delta^{r+1}b_{n+2\nu}$ have a constant sign, the terms $\Delta^r x_n$ also have a constant sign. Moreover, we have by (3) and (9)

$$x_{n+1}+x_n = \sum_{\nu=0}^{p-1}\frac{(-1)^\nu}{2^{\nu+1}}\{\Delta^\nu b_{n+1}+\Delta^\nu b_n\} + \frac{(-1)^p}{2^p}\Delta^p b_n,$$

whence, according to lemma I,

$$x_{n+1}+x_n = b_n.$$

Consequently, the sequence x_n defined by formula (3) actually has all the desired properties. The uniqueness of such a sequence follows from lemma III in view of the fact that condition (2) and the inequality $r \geqslant p$ imply the relation

$$\lim_{n\to\infty}\Delta^r b_n = 0.$$

This completes the proof.

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SUR QUELQUES GÉNÉRALISATIONS DES NOMBRES PSEUDOPREMIERS

PAR

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Soient $a>0$ et $b>0$ des entiers tels que $a>b$ et $(a,b)=1$. Considérons une fonction $f(n)$ à valeurs entières positives, définie pour tout entier $n>0$ et assujettie à la condition

(1) $$(p-1, f(n)) \,|\, f(np)$$

pour tout p premier tel que $p \nmid n$. On a alors les théorèmes suivants:

THÉORÈME 1. *S'il existe un n_0 premier tel que $2<f(n_0)\geqslant n_0$, $n_0 | a^{f(n_0)}-b^{f(n_0)}$ et $f(n)\geqslant n-1$ pour $n>n_0$, il existe aussi, pour tout entier $s>1$, un n composé, produit de s nombres premiers distincts, et tel que*

(2) $$n \,|\, a^{f(n)}-b^{f(n)}.$$

THÉORÈME 2. *S'il existe un n_0 pair tel que $f(n_0)>2$ et $f(n)\geqslant n-1$ pour $n\geqslant n_0$, et qui satisfait à l'une des conditions*

(3) $$n_0 | a^{f(n_0)+1}b-ab^{f(n_0)+1},$$

(4) $$n_0 | a^{f(n_0)}-b^{f(n_0)},$$

il existe aussi une infinité de nombres pairs satisfaisant à (3) ou à (4) respectivement.

On a le théorème (T) suivant [1]:

(T) *Si $a>0$, $b>0$ et $m>2$ sont des entiers tels que $a>b$ et $(a,b)=1$, alors, sauf le cas où $a=2$, $b=1$ et $m=6$, le nombre a^m-b^m a un diviseur p premier (dit primitif) tel que $m|p-1$ et que p ne divise le nombre a^k-b^k pour aucun $k=1,2,\ldots,m-1$.*

LEMME. *Sous les mêmes hypothèses, il existe un p premier tel que*

$$p|a^m-b^m \quad \text{et} \quad m|p-1.$$

[1] Cf. [2], p. 386. Ce théorème a été démontré par Birkhoff et Vandiver [1].