COLLOQUIUM MATHEMATICUM

VOL. IX 1962 FASC. 2

ON ADDITIVE h-BASES FOR n

BY

L. MOSER AND J. RIDDELL (EDMONTON)

A set $A: 0 = a_1 < a_2 < \ldots < a_k$ of real numbers is called an *h-basis for n* if each of $1, 2, \ldots, n$ can be represented as the sum of h elements of A, repetitions being allowed. We denote by $k_h(n)$ the smallest number of elements that can be used to form an h-basis for n. In the following we assume that A is a minimal h-basis for n, that is, that $k = k_h(n)$.

An upper estimate for $k_h(n)$ was found by Rohrbach [4]. He showed by constructing particular bases that, for $h \geqslant 2$,

$$k_h(n) < hn^{1/h}. \tag{1}$$

The case $h = 1$ is trivial. On the other hand, since we can form $\binom{k+h-1}{h}$ h-combinations with repetition of the k elements of our basis, and therefore that many sums, we have

$$\binom{k+h-1}{h} \geqslant n. \tag{2}$$

Using $(k+h-1)^h/h! \geqslant \binom{k+h-1}{h}$ we obtain from (2)

$$\frac{k^h}{h!}(1+\varepsilon) > n \tag{3}$$

for n sufficiently large. Improvements have been made only on the lower estimate and only in the case $h = 2$, when (2) becomes $(k^2+k)/2 > n$ Rohrbach conjectured, in this case, that $k^2 \sim 4n$, and proved

$$\frac{k^2}{2}(1-.0016) > n \tag{4}$$

for n sufficiently large. Ostmann ([2], p. 179) remarks that this result "bereits umfangreiche Betrachtungen erfordert". Then Moser [1], using a simpler argument, proved

$$\frac{k^2}{2}(1-.0194) > n,$$

and by a refinement of this argument Riddell [3] proved

$$\frac{k^2}{2}(1-.0269) > n.$$

For $h > 2$, Stöhr [5] remarks that presumably one can use Rohrbach's method to sharpen the lower estimate for k, and goes on to say "Jedoch werden die diesbezüglichen Rechnungen umfangreich und unübersichtlich". In this note we prove the following theorem by an argument related to that of [1]:

THEOREM. *Let $h \geqslant 3$; then for every ε, $0 < \varepsilon < 1$, and n sufficiently large,*

$$(5) \qquad \frac{k^h}{h!}\left(1-\left(\frac{(1-\varepsilon)\cos\frac{\pi}{h}}{2+\cos\frac{\pi}{h}}\right)^h\right) > n.$$

We remark that (5) is sharper than (3) for $h \geqslant 2$. The following argument does not give (5) if $h = 2$, but for $h = 2$, (5) follows from (4).

Proceeding to the proof of (5) we consider the generating function

$$f(x) = \sum_{j=1}^{k} x^{a_j}$$

and let

$$(6) \qquad g(x) = \frac{f^h(x)}{h!} + f^{h-2}(x)f(x^2).$$

The coefficient of x^j in $g(x)$ will be greater than or equal to the number of representations of j as the sum of h summands from A with order not taken into account. We now define $\delta(f)$ by

$$(7) \qquad g(x) = x+x^2+\ldots+x^n+\sum_{j=0}^{ha_k}\delta(j)x^j.$$

Since A is an h-basis for n it follows that $\delta(j) \geqslant 0$ for all j. For $x = 1$ (7) yields

$$(8) \qquad \frac{k^h}{h!}+k^{h-1} = n+\sum\delta(j).$$

(8) implies $k^h/h!+k^{h-1} \geqslant n$, and while not as sharp as (2), this also implies (3). We obtain the improvement (5) by showing that $\sum\delta(j)$ is large.

Since $x+x^2+\ldots+x^n$ is zero for $x = e^{2\pi i/n}$, we have from (7),

$$g(e^{2\pi i/n}) = \sum\delta(j)e^{2\pi ij/n}.$$

Hence,

$$(9) \qquad \sum\delta(j) \geqslant |g(e^{2\pi i/n})| \geqslant \frac{1}{h!}|f(e^{2\pi i/n})|^h-|f^{h-2}(e^{2\pi i/n})f(e^{4\pi i/n})|$$

$$\geqslant \frac{1}{h!}|f(e^{2\pi i/n})|^h-k^{h-1}.$$

(by the triangle inequality)

We find a lower estimate for the right hand side of (9) by considering for each a_j the vector from the origin to the point $e^{2\pi i a_j/n}$ on the unit circle. The sum of the projections of the k vectors associated with A onto the line from the origin to $f(e^{2\pi i/n})$ is of course $|f(e^{2\pi i/n})|$ and greater than or equal to the sum of the projections of these vectors onto any other line. We let p be the number of elements of A exceeding n/h, and consider the sum of the projections of our k vectors onto the line $\theta = \pi/h$. Noting that the amplitude of the vector associated with a_j is $2\pi a_j/n$, we see that $k-p$ vectors have amplitude θ in the interval $0 \leqslant \theta \leqslant 2\pi/h$ and each contributes at least $\cos\pi/h$ to this sum, while the remaining p vectors each contributes at least -1. Hence, $|f(e^{2\pi i/n})| \geqslant (k-p)\cos\pi/h-p$, and from (9) we obtain

$$(10) \qquad \sum\delta(j) \geqslant \frac{\big(k\cos\pi/h-p(1+\cos\pi/h)\big)^h}{h!}-k^{h-1}.$$

If p is small we shall use (10). On the other hand, since the sums formed using the p a's which exceed n/h are all greater than n, it follows from (6) and (7) that

$$(11) \qquad \sum\delta(j) \geqslant \frac{p^h}{h!}+p^{h-1} \geqslant \frac{p^h}{h!}.$$

We let $\alpha = (\cos\pi/h)/(2+\cos\pi/h)$ and consider the two cases (i) $p < k\alpha$ and (ii) $p \geqslant k\alpha$. In case (i), (10) yields.

$$(12) \qquad \sum\delta(j) \geqslant \frac{k^h\alpha^h}{h!}-k^{h-1},$$

while in case (ii), (12) follows from (11). Hence, from (8) and (12) we obtain

$$(13) \qquad \frac{k^h}{h!}(1-\alpha^h)+2k^{h-1} \geqslant n;$$

and the theorem follows.

Rohrbach considered also restricted 2-bases for n, that is, bases in which no element exceeds $[(n+1)/2]$. For such bases he proved that for

n sufficiently large,

$$\frac{k^2}{2}(1-.0692) > n,$$

and Riddell [3] proved

$$\frac{k^2}{2}(1-.1329) > n.$$

If we define a restricted h-basis for n to be an h-basis for n in which no element exceeds n/h, then we have

$$\frac{k^h}{h!}\left(1-\left((1-\varepsilon)\cos\frac{\pi}{h}\right)^h\right) > n \tag{14}$$

for n sufficiently large. This follows directly from (8) and (10) with $p = 0$.

Notice that Rohrbach himself considered only bases composed of integers. Then we require $[(n+h-1)/h]$ here instead of n/h. Formula (14) still follows, but not simply by putting $p = 0$ in (10).

REFERENCES

[1] L. Moser, *On the representation of* $1, 2, \ldots, n$ *by sums*, Acta Arithmetica 6 (1960), p. 11-13.

[2] H. Ostmann, *Additive Zahlentheorie*, Part I, Berlin—Göttingen—Heidelberg 1956.

[3] J. Riddell, *Master's Thesis*, University of Alberta.

[4] H. Rohrbach, *Ein Beitrag zur additiven Zahlentheorie*, Mathematische Zeitschrift 42 (1937), p. 1-30.

[5] A. Stöhr, *Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe, I*, Journal für die reine und angewandte Mathematik 194 (1955), p. 40-65.

UNIVERSITY OF ALBERTA

Reçu par la Rédaction le 9. 10. 1961

COLLOQUIUM MATHEMATICUM

VOL. IX 1962 FASC. 2

SOLUTION D'UN PROBLÈME DE K. ZARANKIEWICZ SUR LES SUITES DE PUISSANCES CONSÉCUTIVES DE NOMBRES IRRATIONNELS

(*DÉDIÉ À LA MÉMOIRE DE K. ZARANKIEWICZ*)

PAR

A. SCHINZEL (VARSOVIE)

K. Zarankiewicz a posé le problème: existe-il un nombre irrationnel q tel qu'on puisse extraire de la suite $q, q^2, \ldots$ quatre termes formant une progression arithmétique? (cf. [2], p. 44, P 115).

La réponse négative à ce problème (même quand on admet q complexe, les quatre termes correspondants étant distincts) est une conséquence immédiate du théorème 2, qui va suivre.

En appliquant la méthode ingénieuse de W. Ljunggren [1], le théorème suivant sera d'abord établi:

THÉORÈME 1. *Les nombres n et m étant des entiers tels que $n > m > 0$ et $n \neq 2m$, le polynôme*

$$g(x) = \frac{x^n-2x^m+1}{x^{(n,m)}-1}$$

est irréductible, à l'exception des cas $n = 7k$, $m = 2k$ et $n = 7k$, $m = 5k$, dans lesquels $g(x)$ est un produit de deux facteurs irréductibles, à savoir

$$(x^{3k}+x^{2k}-1)(x^{3k}+x^k+1) \quad \text{et} \quad (x^{3k}+x^{2k}+1)(x^{3k}-x^k-1)$$

respectivement.

LEMME. *Soit*

$$f(x) = x^n-2x^m+1 = \varphi_r(x)\psi_s(x) \quad \text{où} \quad r+s = n, \tag{1}$$

$\varphi_r(x)$ et $\psi_s(x)$ étant des polynômes normés de degré r et s respectivement, et aux coefficients entiers. Soit en outre

$$\langle 7k, 2k\rangle \neq \langle n, m\rangle \neq \langle 7k, 5k\rangle.$$

Alors au moins l'un des deux facteurs de (1) *est un polynôme réciproque.*