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Reçu par la Rédaction le 5. 11. 1961

COLLOQUIUM MATHEMATICUM

VOL. IX 1962 FASC. 2

ON THE JOINT LIMITING DISTRIBUTION OF TIMES SPENT IN PARTICULAR STATES BY A MARKOV PROCESS

BY

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We consider a homogeneous separable Markov process $\xi(t)$ with a finite number of possible states $a_0, a_1, \ldots, a_k$. Let $p_{ij}(t)$ be the probabilities of transition from the state a_i to the state a_j in time t $(i, j = 0, 1, \ldots, k)$. We suppose that

1° $\lim\limits_{t\to 0^+} p_{ii}(t) = 1 \quad (i = 0, 1, \ldots, k)$,

2° for each pair i, j $(i, j = 0, 1, \ldots, k)$ there exists t_0 such that $p_{ij}(t_0) > 0$.

It follows from these assumptions that almost all sample functions are step functions and $p_{ij}(t) > 0$ if $t > 0$ (cf. [2], chapter VI).

Denote by $X_i(t)$ the total time spent in the state a_i during the time interval $[0, t]$. Thus

$$X_i(t) = \int_0^t \chi_i(u)\,du \qquad (i = 0, 1, \ldots, k), \tag{1}$$

where $\chi_i(t)$ is the characteristic function of the set $\{t : \xi(t) = a_i\}$ $(i = 0, 1, \ldots, k)$. It is quite clear that $\sum_{i=0}^{k} X_i(t) = t$.

The aim of this paper is to find the limiting k-dimensional cumulative distribution function of the random variable $\{X_1(t), \ldots, X_k(t)\}$ when $t \to \infty$.

The one-dimensional variable $X_i(t)$ has been investigated by Takács [7]—[9] and Rényi [5] (in [8] and [9] the assumptions are more general). In the case of Markov chains the above-mentioned problem was treated by Romanowski [6], p. 233, and Kolmogorov [4].

Since almost all sample functions are step functions, we may define a sequence of random variables $x_i^1, \bar{x}_i^1, x_i^2, \bar{x}_i^2, \ldots$, as follows:

1° if $\xi(0) = a_i$, then

$$x_i^1 = \sup\{t\colon \xi(u) = a_i \text{ for } 0 \leqslant u \leqslant t\},$$
$$\bar{x}_i^1 = \sup\{t\colon \xi(u+x_i^1) \neq a_i^1 \text{ for } 0 \leqslant u \leqslant t\},$$
$$x_i^2 = \sup\{t\colon \xi(u+x_i^1+\bar{x}_i^1) = a_i \text{ for } 0 \leqslant u \leqslant t\},$$
$$\bar{x}_i^2 = \sup\{t\colon \xi(u+x_i^1+\bar{x}_i^1+x_i^2) \neq a_i \text{ for } 0 \leqslant u \leqslant t\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

2° If $\xi(0) \neq a_i$, then

$$\bar{x}_i^1 = \sup\{t\colon \xi(u) \neq a_i \quad \text{for } 0 \leqslant u \leqslant t\},$$
$$x_i^1 = \sup\{t\colon \xi(\bar{x}_i^1+u) = a_i \text{ for } 0 \leqslant u \leqslant t\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Thus $x_i^1, \bar{x}_i^1, x_i^2, \bar{x}_i^2, \ldots$ denote the lengths of intervals on the time axis, in which successively $\xi(t) = a_i, \neq a_i, a_i$, and so on.

Let us write

$$m_i = \mathrm{E}(x_i^j), \quad \overline{m}_i = \mathrm{E}(\bar{x}_i^j), \quad s_i^2 = \mathrm{Var}(x_i^j), \quad \bar{s}_i^2 = \mathrm{Var}(\bar{x}_i^j)$$
$$(i = 0, 1, \ldots, k;\ j = 1, 2, \ldots).$$

The main result of this paper is the following theorem:

THEOREM 1. *Let $X_i(t)$ $(i = 0, 1, \ldots, k)$ be defined by* (1). *Then*

$$\lim_{t=\infty} \mathrm{P}\left\{\frac{X_i(t) - \dfrac{m_i t}{m_i+\overline{m}_i}}{\sqrt{t}} \leqslant c_i;\ i = 1, 2, \ldots, k\right\} \tag{2}$$
$$= \frac{1}{(2\pi)^{k/2}\sqrt{|L|}} \int_{-\infty}^{c_1} \ldots \int_{-\infty}^{c_k} \exp\left\{-\frac{1}{2|L|}\sum_{i,j=1}^{k} |L_{ij}|\, u_i u_j\right\} du, \ldots, du_k,$$

where L_{ij} are cofactors of the elements of the determinant $|l_{ij}|$ and l_{ij} are given by the formulae

$$l_{ij} = \begin{cases} \dfrac{\overline{m}_i^2 s_i^2 + m_i^2 \bar{s}_i^2}{(m_i+\overline{m}_i)^3} & \text{for} \quad i = j, \\[2ex] \dfrac{1}{(m_i+\overline{m}_i)(m_j+\overline{m}_i)}\left[\dfrac{m_i(m_j^2\bar{s}_j^2+\overline{m}_j^2 s_j^2)}{m_j^2-\overline{m}_j^2} + \dfrac{m_j(m_i^2\bar{s}_i^2+\overline{m}_i^2 s_i^2)}{m_i^2-\overline{m}_i^2} + \right. \\[2ex] \qquad \left. + \dfrac{m_i m_j(\overline{m}_i\overline{m}_j - m_i m_j)(s_0^2+\bar{s}_0^2)}{(\overline{m}_i - m_i)(\overline{m}_j - m_j)(\overline{m}_0+m_0)} \right] & \text{for} \quad i \neq j. \end{cases} \tag{3}$$

An analogous formula for the one-dimensional variable has been shown in papers [7] and [5]. We may write it in the following form:

$$\lim_{t=\infty} \mathrm{P}\left\{\frac{X_i(t) - \dfrac{m_i t}{m_i+\overline{m}_i}}{\sqrt{t}} \leqslant c\right\} = \frac{1}{\sqrt{2\pi D_i}} \int_{-\infty}^{c} e^{-x^2/2D_i} dx, \tag{4}$$

where

$$D_i = \frac{m_i^2 \bar{s}_i^2 + \overline{m}_i^2 s_i^2}{(m_i+\overline{m}_i)^3}.$$

This formula has been proved under the condition that the following assumptions are satisfied:

Z_1. The random variables x_i^j $(j = 1, 2, \ldots)$ are non-negative and have the same distribution with finite variance. The random variables $\bar{x}_i^j$ $(j = 1, 2, \ldots)$ are non-negative and have the same distribution with finite variance.

Z_2. All random variables $x_i^j, \bar{x}_i^r$ $(j = 1, 2, \ldots;\ r = 1, 2, \ldots)$ are independent.

An inspection of the proof of the theorem given in paper [5] shows that it remains valid if we replace the assumption Z_2 by the assumption

Z_3. The two-dimensional random variables $(x_i^j, \bar{x}_i^j$, $(j = 1, 2, \ldots,$ are all independent and identically distributed and random variables x_i^j and $\bar{x}_i^j$ are uncorrelated.

In this paper we shall use this fact.

Theorem 1 is a consequence of the following theorem:

THEOREM 2. *If*

(i) $\{u_i^1, \ldots, u_i^k\}$, $i = 1, 2, \ldots$, *is a sequence of k-dimensional independent and identically distributed random variables,*

(ii) $\mathrm{E}(u_i^j) = 0$, $\mathrm{E}(u_i^j u_q^j) = l_{iq}$,

(iii) *$N(t)$ is a positive integer-valued random variable such that $N(t)/t$ is stochastically convergent to a number $c > 0$ as $t \to \infty$,*

(iv) $$z_i^{N(t)} = \sum_{j=1}^{N(t)} u_i^j,$$

then

$$\lim \mathrm{P}\left\{\frac{z_i^{N(t)}}{\sqrt{N(t)}} \leqslant c_i;\ i = 1, 2, \ldots, k\right\}$$
$$= \frac{1}{(2\pi)^{k/2}|L|} \int_{-\infty}^{c_1} \ldots \int_{-\infty}^{c_k} \exp\left\{-\frac{1}{2|L|}\sum L_{iq} u_i u_q\right\} du_1, \ldots, du_k,$$

where L_{iq} are cofactors of the elements of the determinant $|l_{iq}|$.

Proof of theorem 2. For any positive number $\varepsilon > 0$ we can find a t_0 such that for $t > t_0$ we have

$$P\left(\left|\frac{N(t)}{t} - c\right| > \varepsilon c\right) < \varepsilon,$$

whence

$$P\left(\left|\frac{N(t)}{t} - c\right| > \varepsilon c\right) = P\left(\frac{N(t)}{t} - c < -\varepsilon c\right) + P\left(\frac{N(t)}{t} - c > \varepsilon c\right)$$
$$= P(N(t) < ct - \varepsilon ct) + P(N(t) > ct + \varepsilon ct)$$
$$= P(N(t) \leqslant N_1) + P(N(t) > N_2) < \varepsilon,$$

where $N_1 = [ct - \varepsilon ct]$, $N_2 = [ct + \varepsilon ct]$.

In virtue of the total probability theorem we may write

$$P\left(\frac{z_i^{N(t)}}{\sqrt{N(t)}} \leqslant c_i;\ i = 1, 2, \ldots, k\right)$$
$$= \sum_{n=1}^{\infty} P\left(\frac{z_i^n}{\sqrt{n}} \leqslant c_i,\ N(t) = n;\ i = 1, 2, \ldots, k\right)$$
$$= \sum_{n=1}^{N_1} P\left(\frac{z_i^n}{\sqrt{n}} \leqslant c_i,\ N(t) = n;\ i = 1, 2, \ldots, k\right) +$$
$$+ \sum_{n=N_1+1}^{N_2} P\left(\frac{z_i^n}{\sqrt{n}} \leqslant c_i,\ N(t) = n;\ i = 1, 2, \ldots, k\right) +$$
$$+ \sum_{n=N_2+1}^{\infty} P\left(\frac{z_i^n}{\sqrt{n}} \leqslant c_i,\ N(t) = n;\ i = 1, 2, \ldots, k\right).$$

Now we are going to prove that for sufficiently large t the first and the third sums can be as small as we like. We have

$$\sum_{n=1}^{N_1} P\left(\frac{z_i^n}{\sqrt{n}} \leqslant \varepsilon_i,\ N(t) = n;\ i = 1, 2, \ldots, k\right) +$$
$$+ \sum_{n=N_2+1}^{\infty} P\left(\frac{z_i^n}{\sqrt{n}} \leqslant c_i,\ N(t) = n;\ i = 1, 2, \ldots, k\right)$$
$$\leqslant \sum_{n=1}^{N_1} P(N(t) = n) + \sum_{n=N_2+1}^{\infty} P(N(t) = n) = P(N(t) \leqslant N_1) + P(N(t) > N_2) < \varepsilon.$$

Then

$$P\left(\frac{z_i^{N(t)}}{\sqrt{N(t)}} \leqslant c_i,\ i = 1, 2, \ldots, k\right) -$$
$$- \sum_{n=N_1+1}^{N_2} P\left(\frac{z_i^n}{\sqrt{n}} \leqslant c_i;\ N(t) = n;\ i = 1, 2, \ldots, k\right) < \varepsilon.$$

Write

$$\varrho_i = \max_{N_1 < n \leqslant N_2} \left|\sum_{N_1 < j \leqslant n} u_i^j\right|.$$

According to Kolmogorov inequality (cf. [8], p. 202) we have

$$P(\varrho_i \geqslant \sqrt[3]{\varepsilon}\sqrt{N_1}) \leqslant \frac{N_2 - N_1}{N_1 \varepsilon^{2/3}} l_{ii}.$$

It is easy to verify that

$$\frac{N_2 - N_1}{N_1 \varepsilon^{2/3}} \leqslant \frac{ct(1+\varepsilon)}{ct(1-\varepsilon) - 1} \varepsilon^{2/3} - \varepsilon^{2/3} < 4\sqrt[3]{\varepsilon}$$

for $t > 1/\varepsilon c$ and $\varepsilon < \frac{1}{8}$. Then

$$P(\varrho_i \geqslant \sqrt[3]{\varepsilon}\sqrt{N_1}) < 4\sqrt[3]{\varepsilon}\, l_{ii} < \varepsilon_1.$$

Now let us remark that for $N_1 < n \leqslant N_2$ the following inequalities are true:

$$z_i^n \leqslant z_i^{N_1} + \varrho_i, \qquad z_i^{N_1} \leqslant z_i^n + \varrho_i.$$

This last inequality follows from the relations:

$$z_i^n + \varrho_i = z_i^{N_1} + u_i^{N_1+1} + \ldots + u_i^n + \varrho_i \text{ and } u_i^{N_1+1} + \ldots + u_i^n + \varrho_i \geqslant 0.$$

Let A_i denote the event $\varrho_i \leqslant \sqrt[3]{\varepsilon}\sqrt{N_1}$, B the event $N_1 < N(t) \leqslant N_2$, C the event $z_i^{N_1}/\sqrt{N_1} \leqslant c_i\sqrt{(1+\varepsilon)/(1-2\varepsilon)} + \varrho_i/\sqrt{N_1}$ $(i = 1, 2, \ldots, k)$, D the event $z_i^{N_1}/\sqrt{N_1} \leqslant c_i - \sqrt[3]{\varepsilon}$ $(i = 1, 2, \ldots, k)$ and E the event $z_i^{N_1}/\sqrt{N_1} \leqslant c_i - \varrho_i/\sqrt{N_1}$ $(i = 1, 2, \ldots, k)$. Then

$$\sum_{n=N_1+1}^{N_2} P\left(\frac{z_i^n}{\sqrt{n}} \leqslant c_i,\ N(t) = n;\ i = 1, 2, \ldots, k\right)$$
$$\leqslant \sum_{n=N_1+1}^{N_2} P\left(\frac{z_i^{N_1} - \varrho_i}{\sqrt{n}} \leqslant c_i,\ N(t) = n;\ i = 1, 2, \ldots, k\right)$$
$$\leqslant \sum_{n=N_1+1}^{N_2} P\left(\frac{z_i^{N_1} - \varrho_i}{\sqrt{N_2}} \leqslant c_i,\ N(t) = n;\ i = 1, 2, \ldots, k\right)$$

$$= \mathrm{P}\left(z_i^{N_1} \leqslant c_i\sqrt{N_2}+\varrho_i,\ N_1 < N(t) \leqslant N_2;\ i=1,2,\ldots,k\right)$$

$$\leqslant \mathrm{P}\left(z_i^{N_1} \leqslant c_i\sqrt{N_2}+\varrho_i;\ i=1,2,\ldots,k\right)$$

$$\leqslant \mathrm{P}\left(\frac{z_i^{N_1}}{\sqrt{N_1}} \leqslant c_i\sqrt{\frac{N_2}{N_1}}+\frac{\varrho_i}{\sqrt{N_1}};\ i=1,2,\ldots,k\right)$$

$$\leqslant \mathrm{P}\left(\frac{z_i^{N_1}}{\sqrt{N_1}} \leqslant c_i\sqrt{\frac{1+\varepsilon}{1-2\varepsilon}}+\frac{\varrho_i}{\sqrt{N_1}};\ i=1,2,\ldots,k\right) = P(C)$$

$$= \mathrm{P}(CA_1A_2\ldots A_k)+\mathrm{P}(C\bar{A}_1A_2\ldots A_k)+\mathrm{P}(C\bar{A}_2A_3\ldots A_k)+$$
$$+\mathrm{P}(C\bar{A}_3A_4\ldots A_k)+\ldots+\mathrm{P}(C\bar{A}_{k-1}A_k)+\mathrm{P}(C\bar{A}_k)$$

$$\leqslant \mathrm{P}\left(\frac{z_i^{N_1}}{\sqrt{N_1}} \leqslant c_i\sqrt{\frac{1+\varepsilon}{1-2\varepsilon}}+\sqrt[3]{\varepsilon};\ i=1,2,\ldots,k\right)+$$
$$+\mathrm{P}(\bar{A}_1)+\mathrm{P}(\bar{A}_2)+\ldots+\mathrm{P}(\bar{A}_k)$$

$$\leqslant \mathrm{P}\left(\frac{z_i^{N_1}}{\sqrt{N_1}} \leqslant c_i\sqrt{\frac{1+\varepsilon}{1-2\varepsilon}}+\sqrt[3]{\varepsilon};\ i=1,2,\ldots,k\right)+k\cdot\varepsilon_1.$$

Thus we have found the upper estimate; now we shall find the lower estimate. Let us remark that

$$\sum_{n=N_1+1}^{N_2} \mathrm{P}\left(\frac{z_i^n}{\sqrt{n}} \leqslant c_i,\ N(t)=n;\ i=1,2,\ldots,k\right)$$

$$\geqslant \sum_{n=N_1+1}^{N_2} \mathrm{P}\left(\frac{z_i^{N_1}+\varrho_i}{\sqrt{n}} \leqslant c_i,\ N(t)=n;\ i=1,2,\ldots,k\right)$$

$$\geqslant \sum_{n=N_1+1}^{N_2} \mathrm{P}\left(\frac{z_i^{N_1}+\varrho_i}{\sqrt{N_1}} \leqslant c_i,\ N(t)=n;\ i=1,2,\ldots,k\right)$$

$$= \mathrm{P}\left(\frac{z_i^{N_1}}{\sqrt{N_1}} \leqslant c_i-\frac{\varrho_i}{\sqrt{N_1}},\ N_1 < N(t) \leqslant N_2;\ i=1,2,\ldots,k\right)$$

$$= \mathrm{P}(E)-\mathrm{P}(E,N(t)\leqslant N_1)-\mathrm{P}(E,N(t)>N_2)$$
$$\geqslant \mathrm{P}(E)-\mathrm{P}(N(t)\leqslant N_1)-\mathrm{P}(N(t)>N_2) > \mathrm{P}(E)+\varepsilon$$
$$\geqslant \mathrm{P}(EA_1\ldots A_k)+\varepsilon$$

$$\geqslant \mathrm{P}\left(\frac{z_i^{N_1}}{\sqrt{N_1}} \leqslant c_i-\sqrt[3]{\varepsilon},\ A_1\ldots A_k;\ i=1,2,\ldots,k\right)$$

$$= \mathrm{P}(D)-\mathrm{P}(D\overline{A_1\ldots A_k}) \geqslant \mathrm{P}(D)-\mathrm{P}(\overline{A_1\ldots A_k})$$

$$= \mathrm{P}(D)-\mathrm{P}(\bar{A}_1A_2\ldots A_k)-\mathrm{P}(\bar{A}_2A_3\ldots A_k)-\ldots-\mathrm{P}(\bar{A}_k)$$
$$\geqslant \mathrm{P}(D)-\mathrm{P}(\bar{A}_1)-\mathrm{P}(\bar{A}_2)-\ldots-\mathrm{P}(\bar{A}_k)$$
$$\geqslant \mathrm{P}\left(\frac{z_i^{N_1}}{\sqrt{N_1}} \leqslant c_i-\sqrt[3]{\varepsilon};\ i=1,2,\ldots,k\right)-k\varepsilon_1.$$

Finally we have

$$\mathrm{P}\left(\frac{z_i^{N_1}}{\sqrt{N_1}} \leqslant c_i-\sqrt[3]{\varepsilon};\ i=1,2,\ldots,k\right)-k\varepsilon_1$$
$$\leqslant \mathrm{P}\left(\frac{z_i^{N(t)}}{\sqrt{N(t)}} \leqslant c_i;\ i=1,2,\ldots,k\right)$$
$$\leqslant \mathrm{P}\left(\frac{z_i^{N_1}}{\sqrt{N_1}} \leqslant c_i\sqrt{\frac{1+\varepsilon}{1-2\varepsilon}}+\sqrt[3]{\varepsilon};\ i=1,2,\ldots,k\right)+k\varepsilon_1.$$

If $t\to\infty$, then N_1 also tends to infinity.

Now the assertion of theorem 2 immediately follows from the last relation and the central limit theorem.

To prove theorem 1 we need the following lemma:

LEMMA 1. *Suppose that $\eta_1,\eta_2,\ldots$ are independent and identically distributed random variables with mean values m and finite variances σ^2. Further, let $N(t)$ denote a positive integer-valued random variable defined for any $t>0$ such that $N(t)/t$ converges for $t\to\infty$ in probability to a constant $c>0$. Then we have for an arbitrary $\varepsilon>0$*

$$\lim_{t=\infty}\mathrm{P}\left(\left|\frac{\eta_1+\eta_2+\ldots+\eta_{N(t)}}{N(t)}-m\right|<\varepsilon\right)=1.$$

In [5] the following formula has been proved:

$$\lim_{t=\infty}\mathrm{P}\left(\frac{(\eta_1-m)+\ldots(\eta_{N(t)}-m)}{\sqrt{N(t)}}\leqslant c\right)=\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{c}e^{-u^2/2\sigma^2}du.$$

The proof of lemma 1 is analogous to the proof of the above relation and we may omit it.

Now suppose that $\xi(0)=a_0$. (In the case $\xi(0)\neq a_0$ some modifications ought to be introduced to the proof of theorem 1 and lemma 2. These modifications are not essential and we shall not consider each of the cases $\xi(0)=a_0$ and $\xi(0)\neq a_0$) separately.

Now we introduce the following notation:

Let $\tau_0<\tau_1<\tau_2<\ldots$ be points on the real axis such that

$$\tau_0=0,\qquad \tau_i=\sum_{j=1}^{i}x_0^j+\bar{x}_0^j.$$

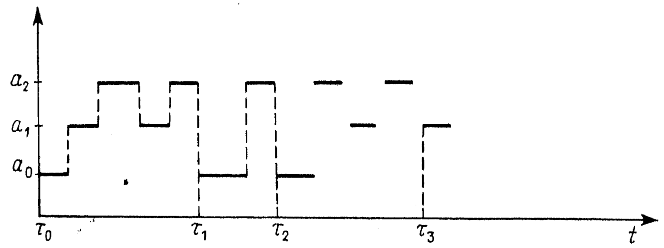
Let y_i^j be random variables such that

(5) $$y_i^j = \int_{\tau_{j-1}}^{\tau_j} \chi_i(u)\,du \qquad (i = 1, 2, \ldots, k;\ j = 1, 2, \ldots)$$

and

(6) $$\bar{y}_i^j = x_0^j + \bar{x}_0^j - y_j^i \qquad (i = 1, 2, \ldots, k;\ j = 1, 2, \ldots).$$

The random variable y_i^j is the time spent in the state a_i between the $(j-1)$-th and j-th appearance of the state a_0.



The case $k = 2$.

From the properties of the process considered it follows that the random variables y_i^j $(j = 1, 2, \ldots)$ are independent and identically distributed.

LEMMA 2. *The moments of the random variables $\bar{x}_i^j, y_i^j, \bar{y}_i^j$ are given by the formulae*

(7) $$\mathrm{E}(\bar{x}_i^j) = \bar{m}_i = -\frac{dL_i(z)}{dz}\bigg|_{z=0},$$

(8) $$\mathrm{Var}(\bar{x}_i^j) = \bar{s}_i^2 = \frac{d^2L_i(z)}{dz^2} - \left[\frac{dL_i(z)}{dz}\right]^2\bigg|_{z=0},$$

(9) $$\mathrm{E}(y_i^j) = \mu_i = \frac{m_i}{m_i+\bar{m}_i}(m_0+\bar{m}_0) < \infty,$$

(10) $$\mathrm{Var}(y_i^j) = \sigma_i^2 = \frac{(m_0+\bar{m}_0)(m_i^2\bar{s}_i^2+\bar{m}_i^2 s_i^2) - m_i^2(m_i+\bar{m}_i)(s_0^2+\bar{s}_0^2)}{(m_i+\bar{m}_i)^2(\bar{m}_i-m_i)},$$

(11) $$\mathrm{E}[(y_i^j-\mu_i)(y_l^j-\mu_l)] = 0 \quad \textit{for} \quad i \neq l,$$

(12) $$\mathrm{E}(\bar{y}_i^j) = \bar{\mu}_i = m_0+\bar{m}_0-\mu_i,$$

(13) $$\mathrm{Var}(\bar{y}_i^j) = \bar{\sigma}_i^2 = s_0^2+\bar{s}_0^2-\sigma_i^2,$$

where

$$L_i(z) = 1+\frac{z}{\lambda_i} - \frac{1}{\lambda_i \int_0^\infty e^{-zt} p_{ii}(t)\,dt}, \qquad \lambda_i = \lim_{t\to 0}\frac{1-p_{ii}(t)}{t}.$$

Proof. The random variable $\bar{x}_i^j$ is the time of return to the state a_i. The mean value and the variance of this variable has been found in [7, § 3.3].

Since

$$\lim_{t=\infty} p_{ij}(t) = \frac{m_i}{m_i+\bar{m}_i} > 0 \quad \text{and} \quad m_i < \infty,$$

we have $\bar{m}_i < \infty$.

Now we shall prove formula (9). We define positive integer-valued random variables $N_i(t)$ by the inequalities:

(14) $$\sum_{j=1}^{N_i(t)} (x_i^j+\bar{x}_i^j) \leqslant t < \sum_{j=1}^{N_i(t)+1} (x_i^j+\bar{x}_i^j) \qquad (i = 0, 1, \ldots, k).$$

It follows from (6) that the following inequalities are also true:

(15) $$\sum_{j=1}^{N_0(t)} (y_i^j+\bar{y}_i^j) \leqslant t < \sum_{j=1}^{N_0(t)+1} (y_i^j+\bar{y}_i^j) \qquad (i = 0, 1, \ldots, k).$$

Representing $X_i(t)$ as the sum of the segments y_i^j, we have the inequalities

$$\frac{\sum_{j=1}^{N_0(t)} y_i^j}{N_0(t)}\cdot\frac{N_0(t)}{t} = \frac{\sum_{j=1}^{N_0(t)} y_i^j}{t} \leqslant \frac{X_i(t)}{t} < \frac{\sum_{j=1}^{N_0(t)+1} y_i^j}{t} = \frac{\sum_{j=1}^{N_0(t)+1} y_i^j}{N_0(t)+1}\cdot\frac{N_0(t)+1}{t}.$$

Taking into account

$$\frac{N_0(t)}{t} \xrightarrow[\text{stochast.}]{} \frac{1}{\mu_i+\bar{\mu}_i}$$

and lemma 1, we have

(16) $$\frac{X_i(t)}{t} \xrightarrow[\text{stochast.}]{} \frac{\mu_i}{\mu_i+\bar{\mu}_i}.$$

On the other hand, we may represent $X_i(t)$ as the sum of the segments x_i^j. Then we may write

$$\frac{\sum_{j=1}^{N_i(t)} x_i^j}{N_i(t)}\cdot\frac{N_i(t)}{t} \leqslant \frac{X_i(t)}{t} < \frac{\sum_{j=1}^{N_i(t)+1} x_i^j}{N_i(t)+1}\cdot\frac{N_i(t)+1}{t}.$$

It follows from lemma 1 and the relation

$$\frac{N_i(t)}{t} \xrightarrow[\text{stochast.}]{} \frac{1}{m_i+\overline{m}_i}$$

that

$$\frac{X_i(t)}{t} \xrightarrow[\text{stochast.}]{} \frac{m_i}{m_i+\overline{m}_i}. \tag{17}$$

Now, comparing (16) and (17), we have

$$\frac{m_i}{m_i+\overline{m}_i} = \frac{\mu_i}{\mu_i+\overline{\mu}_i}.$$

Using the relation $\mu_i+\overline{\mu}_i = m_0+\overline{m}_0$, we obtain

$$\mu_i = \frac{m_i}{m_i+\overline{m}_i}(m_0+\overline{m}_0).$$

Thus formula (9) is proved.

Representing $X_i(t)$ as the sum of the segments y_i^j and using formula (4) (the assumptions Z_1 and Z_3 being satisfied for y_i^j, $\overline{y}_i^j$) we get

$$\lim_{t\to\infty} \mathrm{P}\left(\frac{X_i(t)-\dfrac{\mu_i t}{\mu_i+\overline{\mu}_i}}{\sqrt{t}} \leqslant c\right) = \frac{1}{\sqrt{2\pi B_i}}\int_{-\infty}^{c} e^{-x^2/2B_i}dx, \tag{18}$$

where

$$B_i = \frac{\mu_i^2\overline{\sigma}_i^2+\overline{\mu}_i^2\sigma_i^2}{(\mu_i+\overline{\mu}_i)^3}.$$

Now, representing $X_i(t)$ as the sum of the segments x_i^j and using again formula (4) (the assumptions Z_1 and Z_2 being satisfied), we get

$$\lim_{t=\infty} \mathrm{P}\left(\frac{X_i(t)-\dfrac{m_i t}{m_i+\overline{m}_i}}{\sqrt{t}} \leqslant c\right) = \frac{1}{\sqrt{2\pi D_i}}\int_{-\infty}^{c} e^{-x^2/2D_i}dx, \tag{19}$$

where

$$D_i = \frac{m_i^2\overline{s}_i^2+\overline{m}_i^2 s^2}{(m_i+\overline{m}_i)^3}.$$

The left sides of the formulae (18), (19) being equal, the right sides must also be equal, that is

$$\frac{\mu_i^2\overline{\sigma}_i^2+\overline{\mu}\,\sigma_i^2}{(\mu_i+\overline{\mu}_i)^3} = \frac{m_i^2\overline{s}_i^2+\overline{m}_i^2 s^2}{(m_i+\overline{m}_i)^3}. \tag{20}$$

Now taking into account that $\mu_i+\overline{\mu}_i = m_0+\overline{m}_0$ and finding σ_i^2 from formula (20), we get

$$\sigma_i^2 = \frac{(m_0+\overline{m}_0)(m_i^2\overline{s}_i^2+\overline{m}_i^2 s_i^2)-m_i^2(m_i+\overline{m}_i)(s_0^2+\overline{s}_0^2)}{(m_i+\overline{m}_i)^2(\overline{m}-m_i)}.$$

We have thus proved formula (10).

Now we are going to show that the random variables y_i^j and y_l^j are uncorrelated for $i\neq l$.

Let us denote by $h_{il}(u,v)$ the frequency function of the two-dimensional random variable (y_i^j, y_l^j) and by $q_0(r,s)$ — the probability of the appearance of the state a_i r times and of the state a_l s times between two successive appearances of the state a_0. Then

$$h_{il}(u,v) = \sum_{r,s} q_0(r,s)\,g_i(u)\,g_l(v), \tag{21}$$

where

$$g_i(u) = \frac{d}{du}P(y_i^j \leqslant u), \quad g_l(v) = \frac{d}{dv}P(y_l^i \leqslant v).$$

From formula (21) it follows that

$$E[(y_i^j-\mu_i)(y_l^j-\mu_l)]$$
$$= \int_0^\infty\int_0^\infty \sum_{r,s} q_0(r,s)[(u-\mu_i)g_i(u)][(v-\mu_l)g_l(v)]\,du\,dv$$
$$= \sum_{r,s} q_0(r,s)\int_0^\infty (u-\mu_i)g_i(u)\,du\int_0^\infty (v-\mu_l)g_l(v)\,dv = 0.$$

We have thus proved formula (11).

Formulae (12) and (13) immediately follow from (6) and (11).

Proof of the theorem 1. It follows from the previous considerations that

$$\lim_{t=\infty}\mathrm{P}\left(\frac{\left(X_i(t)-\dfrac{m_i}{m_i+\overline{m}_i}t\right.}{\sqrt{t}} \leqslant c_i;\ i = 1,2,\ldots,k\right) \tag{22}$$
$$= \lim_{t=\infty}\mathrm{P}\left(\frac{X_i(t)-\dfrac{\mu_i}{\mu_i+\overline{\mu}_i}t}{\sqrt{t}} \leqslant c_i;\ i = 1,2,\ldots,k\right).$$

Let us remark that we may estimate $X_i(t)$ in the following way:

$$\sum_{j=1}^{N_0(t)} y_i^j \leqslant X_i(t) < \sum_{j=1}^{N_0(t)+1} y_i^j, \tag{23}$$

where $N_0(t)$ is defined by (15).

In view of (15) and (23)

$$(24)\qquad \sum_{j=1}^{N_0(t)} y_i^j - \frac{\mu_i}{\mu_i+\bar{\mu}_i}\sum_{j=1}^{N_0(t)+1}(y_i^j+\bar{y}_i^j) \leqslant X_i(t) - \frac{\mu_i}{\mu_i+\bar{\mu}_i}t$$

$$\leqslant \sum_{j=1}^{N_0(t)+1} y_i^j - \frac{\mu_i}{\mu_i+\bar{\mu}_i}\sum_{j=1}^{N_0(t)}(y_i^j+\bar{y}_i^j).$$

Write

$$\mu_i+\bar{\mu}_i = m_0+\bar{m}_0 = M$$

and

$$\alpha_i^j = \frac{1}{M^{3/2}}[My_i^j - \mu_i(y_i^j+\bar{y}_i^j)] = \frac{1}{M^{3/2}}[\bar{\mu}_i y_i^j - \mu_i \bar{y}_i^j].$$

Then inequality (24) can be expressed as follows:

$$(25)\quad \sum_{j=1}^{N_0(t)} \alpha_i^j\sqrt{m} - \frac{\mu_i}{M}(y_i^{N_0(t)+1}+\bar{y}_i^{N_0(t)+1}) \leqslant X_i(t) - \frac{\mu_i t}{M} \leqslant \sum_{j=1}^{N_0(t)} \alpha_i^j\sqrt{m} + y_i^{N_0(t)+1}.$$

Now we are going to find the moments of the random variables α_i^j. The mean value is equal to

$$\mathrm{E}(\alpha_i^j) = \frac{1}{M^{3/2}}[\bar{\mu}_i\mathrm{E}(y_i^j) - \mu_i\mathrm{E}(\bar{y}_i^j)] = 0.$$

In the same way as we have proved formula (11), we may show that

$$\mathrm{E}[(y_i^j-\mu_i)(\bar{y}_i^j-\bar{\mu}_i)] = \mathrm{E}[(y_i^j-\mu_i)(y_1^j-\mu_1+$$
$$\ldots+y_{i-1}^j-\mu_{i-1}+y_{i+1}^j-\mu_{i+1}+\ldots+y_k^j-\mu_k+x_0^j-m_0)] = 0.$$

From the last relation and the definition of the random variable α_i^j it follows that

$$\mathrm{Var}(\alpha_i^j) = \frac{\bar{\mu}_i^2\sigma_i^2+\mu_i^2\bar{\sigma}_i^2}{M^3}.$$

In view of (20) the last relation can be written in the following form:

$$\mathrm{Var}\,\alpha_i^j = \frac{\bar{m}_i^2 s_i^2+m_i^2\bar{s}_i^2}{(m_i+\bar{m}_i)^3}.$$

Now, taking into consideration that

$$y_i^j+\bar{y}_i^j = y_l^j+\bar{y}_l^j = x_0^j+\bar{x}_0^j,$$

we have

$$\mathrm{E}(\alpha_i^j\alpha_l^j) = \frac{1}{M^3}\mathrm{E}[My_i^j-\mu_i(y_i^j+\bar{y}_i^j)]\cdot[My_l^j-\mu_l(y_l^j+\bar{y}_l^j)]$$

$$= \frac{1}{M^3}\{M^2\mu_i\mu_l - M\mu_i\mathrm{E}[y_l^j(y_l^j+\bar{y}_l^j)] - M\mu_l\mathrm{E}[y_i^j(y_i^j+\bar{y}_i^j)] + \mu_i\mu_l\mathrm{E}(x_0^j+\bar{x}_0^j)^2\}+$$

$$+\frac{2}{M^3}\mu_i\mu_l M\mathrm{E}(x_0^j+\bar{x}_0^j) + \frac{2}{M^3}\mu_i\mu_l M\mathrm{E}(x_0^j+\bar{x}_0^j)$$

$$= \frac{1}{M^3}\mu_i\mu_l\mathrm{E}(x_3^j+\bar{x}_0^j-M)^2 + \frac{1}{M^2}[\mu_i^2\mu_l+\mu_i\mu_l^2-\mu_i\mathrm{E}(y_l^{j2})-\mu_l\mathrm{E}(y_i^{j2})]$$

$$= \frac{1}{M^3}[\mu_i\mu_l(s_0^2+\bar{s}_0^2) - M(\mu_l\sigma_i^2+\mu_i\sigma_l^2)].$$

In view of (9), (10) $\mathrm{E}[\alpha_i^j\alpha_l^j]$ can be written in the following form:

$$\mathrm{E}(\alpha_i^j\alpha_l^j) = \frac{1}{(m_i+\bar{m}_i)(m_l+\bar{m}_l)}\Bigg[\frac{m_i(m_l^k\bar{s}_l^2+\bar{m}_l^2 s_l^2)}{m_l^2-\bar{m}_l^2} + \frac{m_l(m_i^2\bar{s}_i^2+\bar{m}_i^2 s_i^2)}{m_i^2-\bar{m}_i^2}+$$
$$+\frac{(\bar{m}_i\bar{m}_l-m_im_l)(s_0^2+\bar{s}_0^2)}{(\bar{m}_i-m_i)(\bar{m}_j-m_j)(\bar{m}_0+m_0)}\Bigg].$$

From (25) it follows that

$$\mathrm{P}\left(\frac{\sum_{j=1}^{N_0(t)}\alpha_i^j}{\sqrt{N_0(t)}}\cdot\sqrt{\frac{N_0(t)}{t}M} + \frac{y_i^{N_0(t)+1}}{\sqrt{t}} \leqslant c_i;\ i = 1, 2, \ldots, k\right)$$

$$\leqslant \mathrm{P}\left(\frac{X_i(t)-\dfrac{\mu_i t}{\mu_i+\bar{\mu}_i}}{\sqrt{t}} \leqslant c_i;\ i = 1, 2, \ldots, k\right)$$

$$\leqslant \mathrm{P}\left(\frac{\sum_{j=1}^{N_0(t)}\alpha_i^j}{\sqrt{N_0(t)}}\cdot\sqrt{\frac{N_0(t)}{t}M} - \frac{\mu_i}{M}\,\frac{y_i^{N_0(t)+1}+\bar{y}_i^{N_0(t)+1}}{\sqrt{t}} \leqslant c_i;\ i = 1, 2, \ldots, k\right).$$

The random variable

$$\frac{N_0(t)}{t}M \xrightarrow[\text{stochast.}]{} 1.$$

The mean values and the variances of y_i^j and $\bar{y}_i^j$ are finite, and so $y_i^j/\sqrt{t}$ and $\bar{y}_i^j/\sqrt{t}$ are stochastically convergent to 0. The random variable α_i^j satisfies the assumptions of theorem 2. The assertion of theorem 1 follows immediately from lemma 2, theorem 2 and the last inequality.

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Reçu par la Rédaction le 19. 9. 1961

COLLOQUIUM MATHEMATICUM

VOL. IX 1962 FASC. 2

MIECZYSŁAW BIERNACKI

(30. III. 1891—21. XI. 1959)

Né le 30 mars 1891 à Lublin, fils d'un médecin, Mieczysław Biernacki y passa en 1909 son baccalauréat et, après deux ans d'études de la chimie à l'Université de Cracovie, prit l'inscription à la Sorbonne comme étudiant des mathématiques. Depuis 1914 jusqu'à 1919 il participa comme volontaire aux luttes de l'armée française et puis de la formation polonaise du général J. Haller contre les Allemands au front occidental. Intoxiqué par gaz asphyxiant et grièvement blessé, il retourna, aussitôt rétabli, sur le front français et, démobilisé en 1919, rentra dans son pays avec l'armée Haller.

Dès 1921 Biernacki poursuivit à la Sorbonne ses études mathématiques, y passa sa licence et y soutint le 11 mai 1928 sa thèse du doctorat auprès de la Commission d'examen présidée par le professeur Paul Montel.

Retourné en Pologne, il fut nommé en 1928 assistant à l'Université de Wilno, en 1929 professeur extraordinaire des mathématiques à l'Université de Poznań et en 1937 professeur ordinaire à la même université. Il n'arrêta pas ses recherches scientifiques pendant les terribles années 1939-1944 de l'occupation allemande en Pologne. Il les passa à Lublin, où quelques leçons privées qu'il y donnait lui ont permis de survivre.

En 1944 il devint professeur ordinaire des mathématiques à l'Université Marie Skłodowska-Curie à Lublin et prit une part particulièrement active à l'organisation de cette université. Il y enseignait jusqu'à la fin de ses jours.

Ses mérites publiques et scientifiques lui ont valu des distinctions et des prix. En 1936 la France lui attribua la Croix d'Officier de la Légion d'Honneur; en 1950 il a été décoré par la République Polonaise de la Croix d'or de l'Ordre du Mérite, en 1954 de celle du Chevalier et en 1959 du Commandeur de l'Ordre *Polonia Restituta*. En 1950 il a reçu le Prix scientifique de l'Etat (II^me^ degré). Deux fois la Société Polonaise