REMARKS ON THE EXISTENCE OF THE RIEMANN-STIELTJES INTEGRAL

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In this note are given some conditions of the existence of Riemann-Stieltjes integrals of the form

$$\int_a^b \Phi(f(x))\,df(x). \tag{1}$$

In particular, we prove that if for a fixed function Φ the integral (1) exists for every function f continuous in the interval $\langle a, b\rangle$, then $\Phi=$ const. This implies that the integral $\int_a^b f(x)\,df(x)$ does not exist for every continuous function f. This is the answer to a question proposed by C. Ryll-Nardzewski. Obviously, if the integral exists, it is equal to $\frac{1}{2}(f^2(b)-f^2(a))$.

Let Φ be a function defined on the whole real axis, and f a function defined on the interval $\langle a, b\rangle$. It immediately follows from the definition of the Riemann-Stieltjes integral that if the integral (1) exists, then for each $\varepsilon>0$ there is a $\delta>0$ such that

$$\Big|\sum_{i=1}^{n}\big[\Phi(f(\eta_i))-\Phi(f(\bar\eta_i))\big]\big[f(x_i)-f(x_{i-1})\big]\Big|<\varepsilon \tag{2}$$

for every division of the interval $\langle a, b\rangle$ by the points $a=x_0<x_1<\ldots<x_n=b$ such that

$$\max_{1\leqslant i\leqslant n}|x_i-x_{i-1}|<\delta$$

and for any sequences $\{\eta_i\}$, $\{\bar\eta_i\}$ such that $\eta_i, \bar\eta_i\in\langle x_{i-1}, x_i\rangle$.

Especially

$$\Big|\sum_{i=1}^{n}\big[\Phi(f(x_i))-\Phi(f(x_{i-1}))\big]\big[f(x_i)-f(x_{i-1})\big]\Big|<\varepsilon. \tag{3}$$

We do not know if the implications (3) $\to$ (2) $\to$ (1) hold in general. Subject to certain additional assumptions, the positive answer is given by the following theorems.

THEOREM 1. *If Φ is a monotonic function on the set $f(\langle a, b\rangle)$ and f is a continuous function, then* (3) $\to$ (2).

Proof. We may assume that Φ is a non decreasing function. Let us denote by ξ_i and $\bar\xi_i$ such numbers that

$$f(\xi_i)=\max_{x_{i-1}\leqslant x\leqslant x_i} f(x),\qquad f(\bar\xi_i)=\min_{x_{i-1}\leqslant x\leqslant x_i} f(x).$$

For any term of the sum (2) we have

$$\big|\Phi(f(\eta_i))-\Phi(f(\bar\eta_i))\big|\big[f(x_i)-f(x_{i-1})\big]\leqslant\big|\Phi(f(\xi_i))-\Phi(f(\bar\xi_i))\big|\big[f(\xi_i)-f(\bar\xi_i)\big].$$

It follows from this inequality that the sum (2) may be estimated by a sum of the form (3) relative to the division $a=x_0$, $\min(\xi_1,\bar\xi_1)$, $\max(\xi_1,\bar\xi_1)$, x_1, $\min(\xi_2,\bar\xi_2)$, $\ldots$, $x_n=b$, q. e. d.

THEOREM 2. *If f is continuous on $\langle a, b\rangle$* and *Φ is continuous on $f(\langle a, b\rangle)$ then* (2) $\to$ (1).

Proof. By virtue of the mean value theorem we have

$$\int_{f(x_1)}^{f(x_2)}\Phi(u)\,du=\Phi(y)[f(x_2)-f(x_1)]=\Phi(f(\eta))[f(x_2)-f(x_1)], \tag{4}$$

where $\eta\in\langle x_1, x_2\rangle$. Obviously $y\in\langle f(x_1), f(x_2)\rangle$. Since the function f is continuous on $\langle x_1, x_2\rangle\subset\langle a, b\rangle$, there exists $\bar\eta\in\langle x_1, x_2\rangle$ such that $f(\bar\eta)=y$.

Using (4) we obtain $f(b)$,

$$\sum_{i=1}^{n}\Phi(f(\eta_i))[f(x_i)-f(x_{i-1})]-\int_{f(a)}^{f(b)}\Phi(u)\,du$$
$$=\sum_{i=1}^{n}\Phi(f(\eta_i))[f(x_i)-f(x_{i-1})]-\sum_{i=1}^{n}\int_{f(x_{i-1})}^{f(x_i)}\Phi(u)\,du$$
$$=\sum_{i=1}^{n}\Phi(f(\eta_i))[f(x_i)-f(x_{i-1})]-\sum_{i=1}^{n}\Phi(f(\bar\eta_i))[f(x_i)-f(x_{i-1})]$$
$$=\sum_{i=1}^{n}\big[\Phi(f(\eta_i))-\Phi(f(\bar\eta_i))\big][f(x_i)-f(x_{i-1})].$$

It is evident that if condition (2) holds, the integral (1) exists and is equal to $\int_{f(a)}^{f(b)}\Phi(u)\,du$.

COROLLARY. *The integral*

$$\int_a^b f(x)\,df(x)$$

exists if and only if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for every division $a = x_0 < x_1 < \ldots < x_n = b$ with

$$\max_{1\leqslant i\leqslant n} |x_i - x_{i-1}| < \delta$$

we have

$$\sum_{i=1}^{n} \big(f(x_i)-f(x_{i-1})\big)^2 < \varepsilon. \tag{5}$$

This follows from the previous considerations and from the remark that if condition (5) is fulfilled then the function f is continuous.

We shall also prove

THEOREM 3. *Let Φ be a function defined on the whole real axis. If the integral* (1) *exists for every function f continuous in the interval $\langle a, b\rangle$, then* $\Phi = \text{const}$.

Proof. We shall prove that for each $y \in (-\infty, \infty)$ we have $\Phi'(y) = 0$. Suppose that for a certain y_0 it is not so, *i. e.*, that there exists a sequence $y_n \to y_0$, $y_n \neq y_0$ for $n = 1, 2, \ldots$ such that

$$\left|\frac{\Phi(y_n)-\Phi(y_0)}{y_n - y_0}\right| \geqslant c > 0. \tag{6}$$

Without loss of generality we may assume also that

$$\sum |y_n - y_0|^2 = +\infty, \quad y_n > y_0 \quad\text{and}\quad \Phi(y_n) > \Phi(y_0), \qquad n = 1, 2, \ldots \tag{7}$$

Now we define the function f. We put

$$f(a) = f\left(a+\frac{b-a}{2^{2n}}\right) = y_0, \quad n = 0, 1, 2, \ldots, \qquad f\left(a+\frac{b-a}{2^{2n+1}}\right) = y_n,$$

and we extend f linearly on the intervals

$$\left\langle a+\frac{b-a}{2^{n+1}}, a+\frac{b-a}{2^n}\right\rangle, \qquad n = 0, 1, \ldots$$

The function f is continuous and therefore by virtue of the assumption the integral exists. Now, by virtue of (3) there exists a $\delta > 0$ such that for each division of the interval $\langle a, b\rangle$ by the points $a = \xi_0 < \xi_1 < \ldots < \xi_n = b$ with $[\xi_i - \xi_{i-1}] < \delta$ we have

$$\sum_{i=1}^{n} \big[\Phi\big(f(\xi_i)\big) - \Phi\big(f(\xi_{i-1})\big)\big]\big[f(\xi_i)-f(\xi_{i-1})\big] < 1. \tag{8}$$

Let k be a positive integer for which $(b-a)/2^k < \delta$, and let $l > k$ be a positive integer such that

$$c\sum_{n=k}^{l} |y_n - y_0|^2 > 1 + \sum_{m=2}^{2^k}\left[\Phi\left(f\left(a+\frac{b-a}{2^k}m\right)\right) - \Phi\left(f\left(a+\frac{b-a}{2^k}(m-1)\right)\right)\right]\times \tag{9}$$
$$\times\left[f\left(a+\frac{b-a}{2^k}m\right) - f\left(a-\frac{b-a}{2^k}(m-1)\right)\right].$$

Let us consider the division $a = x_0 < x_1 < \ldots < x_r = b$, where $r = 2l+2^k-k+1$, $x_i = a+(b-a)/2^{2l+2-i}$ for $i = 1, 2, \ldots, 2l+2-k$, $x_i = [a+(b-a)/2^k](i-2l+k-1)$ for $i = 2l+3-k, \ldots, 2l+2^k-k$. In view of (6), (7) and the definition of the function f we have, by virtue of (9),

$$\sum_{i=1}^{2l+2^k-k+1} \big[\Phi\big(f(x_i)\big)-\Phi\big(f(x_{i-1})\big)\big]\big[f(x_i)-f(x_{i-1})\big] \tag{10}$$
$$= \sum_{i=1}^{2l+2-k} \big[\Phi\big(f(x_i)\big)-\Phi\big(f(x_{i-1})\big)\big]\big[f(x_i)-f(x_{i-1})\big] +$$
$$+ \sum_{i=2l+3-k}^{2l+2^k+1-k} \big[\Phi\big(f(x_i)\big)-\Phi\big(f(x_{i-1})\big)\big]\big[f(x_i)-f(x_{i-1})\big] \geqslant 2c\sum_{n=k}^{l}|y_n-y_0|^2 -$$
$$-\sum_{m=2}^{2^k}\left[\Phi\left(f\left(a+\frac{b-a}{2^k}m\right)\right) - \Phi\left(f\left(a+\frac{b-a}{2^k}(m-1)\right)\right)\right]\times$$
$$\times\left[f\left(a+\frac{b-a}{2^k}m\right) - f\left(a+\frac{b-a}{2^k}(m-1)\right)\right] \geqslant 1,$$

which contradicts (8).

Hence assumption (6) leads to a contradiction and therefore $\Phi'(y_0) = 0$, q. e. d.

Reçu par la Rédaction le 18. 6. 1956