On a d'abord pour ξ_1 par l'application itérée de (2) et (v):

(9) $$[(xy)(zu)] = \{[x(zu)][y(zu)]\} = \{[(xz)(xu)][(yz)(yu)]\} = \\ = \{[(xz)(ux)][(zy)(yu)]\},$$

(10) $$[(xy)(zu)] = \{[(xy)z][(xy)u]\} = \{[(xz)(yz)][(xu)(yu)]\} = \\ = \{[(xz)(zy)][(ux)(yu)]\} = (\{(xz)[(ux)(yu)]\}\,\{(zy)[(ux)(yu)]\}) = \\ = (\{[(xz)(ux)][(xz)(yu)]\}\,\{[(zy)(ux)][(zy)(yu)]\}).$$

On a de même:

(11) $$[(zy)(ux)] = \{[(zy)u][(zy)x]\} = \{[(zy)u][x(yz)]\} = \\ = \{[(zu)(yu)][(xy)(xz)]\},$$

(12) $$[(zy)(ux)] = \{[z(ux)][y(ux)]\} = \{[(zu)(zx)][(yu)(yx)]\} = \\ = \{[(zu)(xz)][(yu)(xy)]\} = (\{[(zu)(xz)](yu)\}\,\{(xy)[(zu)(xz)]\}) = \\ = (\{[(zu)(yu)][(xz)(yu)]\}\,\{[(xy)(zu)][(xy)(xz)]\}).$$

Ainsi, en posant

$$\alpha = [(xz)(ux)], \qquad \beta = [(zy)(yu)],$$

$$\xi_3 = [(zy)(ux)],$$

$$\gamma = [(xy)(xz)], \qquad \delta = [(zu)(yu)],$$

les formules (9)-(12) deviennent d'après (8)

$$\xi_1 = (\alpha\beta) = [(\alpha\xi_2)(\xi_3\beta)], \qquad \xi_3 = (\delta\gamma) = [(\delta\xi_2)(\xi_1\gamma)].$$

Reste à appliquer (v) aux couples $\alpha\xi_2$, $\delta\gamma$, $\delta\xi_2$ et $(\delta\xi_2)(\xi_1\gamma)$ pour que ces formules prennent exactement la forme des prémisses de (5), c. q. f. d.

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A NOTE ON A THEOREM OF HELSON

BY

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Helson has proved the following theorem [1]):

If $\circ$ is a group operator on the subsets of a set M with zero the empty set, invariant under simple transformations, and such that $A\circ B\subset A+B$ for all subsets A,B of M, then $\circ$ is symmetric difference.

The purpose of this note is to strengthen his result by dropping the requirement about zero as the empty set and the invariance of $\circ$ under simple transformations.

Theorem. If the subsets of a set M form a group under the operation $\circ$, and if $A\circ B\subset A+B$ for all subsets A,B of M, then $\circ$ is symmetric difference.

I give the proof in six steps.

1⁰ *The empty set 0 is the identity element of the group.*

In fact, the relation $0\circ 0\subset 0+0=0$ implies $0\circ 0=0$.

2⁰ *If A is a one-element set, then $A^{-1}=A$.*

If A is a one-element set, the relation $A\circ A\subset A$ implies either $A\circ A=0$ or $A\circ A=A$. The latter is impossible since A is not the identity.

3⁰ *If A is a one-element set and B is any set, then*

$$A\circ B = B\circ A = A \dot{-} B,$$

where the operator $\dot{-}$ denotes symmetric difference.

Assume to begin with that A *non* $\subset B$.

Now $B=A\circ A\circ B\subset A+(A\circ B)$, hence $B\subset A\circ B$. Since A is not the identity element of the group, this inclusion is proper. Since A has only one element, one of the two inclusions $B\subset A\circ B\subset A+B$ must be improper, hence $A\circ B=A+B$.

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[1]) Henry Helson, *On the symmetric difference of sets as a group operation*, Colloquium Mathematicum 1 (1948), p. 203-205.

Now if $A \subset B$, let $C = B - A$. As we have just shown, $A \circ C = A + C = B$ and $A \circ B = A \circ A \circ C = C = B - A$.

In any case $A \circ B = A \dot{-} B$. A similar argument shows that $B \circ A = A \dot{-} B$.

4⁰ *If A is any set, then $A^{-1} = A$.*

It is sufficient to show that $A^{-1} \subset A$. Suppose not. Choose $p \in A^{-1} - A$. Then $A \circ (p) = A + (p)$ and $A^{-1} - (p) = A^{-1} \circ (p) = [(p) \circ A]^{-1} = [A + (p)]^{-1}$. Furthermore we have $[A + (p)]^2 = [A \circ (p)] \circ [(p) \circ A] = A^2$.

Now for any set B we have $B = B^{-1} \circ B^2 \subset B^{-1} + B^2$. In this relation put $B = A + (p)$. We obtain

$$A + (p) \subset [A + (p)]^{-1} + [A + (p)]^2 = [A^{-1} - (p)] + A^2.$$

Since $A^2 \subset A$ and $p \operatorname{non} \in A$, we find that p appears in the set on the left of this relation but not in the set on the right: a contradiction.

5⁰ *If A and B are disjoint, then $A \circ B = A + B = B \circ A$.*

$B = A \circ A \circ B \subset A + (A \circ B)$, hence $B \subset A \circ B$. Similarly, $A \subset A \circ B$. Hence $A + B \subset A \circ B \subset A + B$. In the same way $B \circ A = A + B$.

6⁰ *If A and B are any two sets, then $A \circ B = A \dot{-} B$.*

$$A \circ B = [(A - B) \circ AB] \circ [AB \circ (B - A)] = (A - B) \circ (B - A) = \\ = (A - B) + (B - A) = A \dot{-} B.$$

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ON A PROBLEM OF SIKORSKI

BY

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Sikorski has posed the following problem [1]):

For each $\gamma < \omega_\mu$ let Z_γ be a family of sequences composed of zeros and ones, each of ordinal type γ; and suppose, for any $\delta < \gamma$, Z_δ consists exactly of those sequences which are segments of type δ of the sequences of Z_γ. Does it follow that there is a family Z of sequences of type ω_μ, such that every Z_γ consists of the segments of type γ of the sequences belonging to Z?

The answer is evidently positive if ω_μ is the limit of a sequence of ordinals of type ω, which implies that μ is a limit ordinal.

We shall show that the answer is negative for every ω_μ smaller than the first regular initial ordinal whose index is a limit ordinal, unless ω_μ is the limit of a denumerable sequence; and that the answer is negative whenever μ is not a limit ordinal.

The proof consists in constructing a counter-example.

First suppose μ is not a limit ordinal, and write $\mu = \tau + 1$. Consider the sequences $A'_\beta = \{\varrho_\gamma\}_{\gamma < \beta}$ of type β (for an arbitrary $\beta < \omega_\mu$), composed of non-zero ordinals smaller than ω_τ, such that no ordinal appears twice in A'_β, and such that the ordinals smaller than ω_τ which do *not* appear form themselves a sequence of type ω_τ. For each $\beta < \omega_\mu$ denote the set of all such sequences by Z'_β. Then for any $\alpha < \beta$, Z'_α consists exactly of the segments of type α of the sequences of Z'_β.

For each sequence A'_β of the sort defined we shall construct a sequence $A_{\nu(\beta)} = \{a_\gamma\}_{\gamma < \nu(\beta)}$ of zeros and ones, of type $\nu(\beta)$, where for each $\alpha \leqslant \beta$ we set $\nu(\alpha) = \sum_{\gamma < \alpha} \varrho_\gamma$. Take $a_0 = 1$, and adjoin a sequence of zeros of type $(-1 + \varrho_0)$, where $(-1 + \varrho_0)$ is the unique ordinal such that $1 + (-1 + \varrho_0) = \varrho_0$, thus defining a_γ for $\gamma < \varrho_0$, all zero except the first. Continue by adjoining a sequence of

[1]) See Colloquium Mathematicum 1 (1948), p. 35, P 19.