Soit maintenant $\mathfrak{a} = \lim\limits_{\xi < \omega_\sigma < \omega_\nu} \mathfrak{a}_\xi$; $\mathfrak{a} \epsilon D_\alpha$ est univoquement determiné par les suites $\mathfrak{a}_\xi \epsilon D_\xi$ dont $\mathfrak{a}$ est un prolongement, d'où $\overline{\overline{D}}_\alpha \leqslant \aleph_\nu^{\aleph_\sigma}$ $(\aleph_\sigma < \aleph_\nu)$. Si $\aleph_\nu = \aleph_{\tau+1}$, on a $\aleph_{\tau+1}^{\aleph_\sigma} = 2^{\aleph_\tau \cdot \aleph_\sigma} = 2^{\aleph_\tau} = \aleph_{\tau+1}$; pour les nombres inaccesibles, l'égalité $\aleph_\nu^{\aleph_\sigma} = \aleph_\nu$ $(\aleph_\sigma < \aleph_\nu)$ se déduit d'un théorème de Tarski[6]), et peut même être établie sans l'hypothèse $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, lorsque ω_ν est inaccessible au sens étroit[7]).

[6]) A. Tarski, *Quelques théorèmes sur les alephs*, Fundamenta Mathematicae 7 (1925), p. 1-14, théorème 7 (p. 7).

[7]) A. Tarski, *Über unerreichbare Kardinalzahlen*, Fundamenta Mathematicae 25 (1935), p. 68-89.

REMARKS ON MEASURE AND CATEGORY

BY

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We call a *measure* every σ-additive set function $\mu(X)$, such that $0 \leqslant \mu(X) \leqslant \infty$, defined in a σ-additive field of subsets of a set $\mathscr{X}$. A measure μ is said to be *σ-finite if* $\mathscr{X}$ is the sum of an enumerable sequence of sets of finite measure μ.

A measure μ defined on the field of all Borel subsets of a topological space[1]) $\mathscr{X}$ is called a *Borel measure* in $\mathscr{X}$.

In this paper $\mathscr{X}$ always denotes a topological space, and μ a Borel measure in $\mathscr{X}$. Our chief problem is the existence of the[2]) decomposition:

(*) $\mathscr{X} = H + K$, where $\mu(H) = 0$ and K is of the first category[3]) in $\mathscr{X}$.

We shall show in a very simple way that the decomposition (*) exists e. g. for α-dimensional measures in separable metric spaces (theorem (iii)), in particular for the linear measure in the plane.

It results from one of our theorems on measures in non-separable spaces that the decomposition (*) is possible for any σ-finite Borel measure vanishing for all one-point sets[3]) whenever $\mathscr{X}$ is a metric space containing a dense subset of potency less than the first innaccessible aleph (theorem (vi)).

[1]) A space is called *topological* if it satisfies the well-known axioms of Kuratowski. See [2], p. 20.

[2]) Since every set of the first category is contained in a set F_σ of the first category, we may always assume (whenever the decomposition (*) is possible) that K is an F_σ and H is a G_δ.

[3]) Our problem can be reduced to the case of measures vanishing for every one-point set. In fact, let X_0 be the set of all x with $\mu((x)) > 0$. If X_0 contains an isolated point of the space, the decomposition (*) is impossible. If X_0 contains no isolated point (i. e. X_0 is of the first category), the decomposition (*) of the space is possible if and only if there exists an analogous decomposition for the measure $\nu(X) = \mu(X - X_0)$ which is a Borel measure in the complementary set of X_0 and vanishes for every one-point set.

This result does not hold for general topological spaces. More precisely, there exists a normal bicompact and totally disconnected space $\mathscr{X}$, and a finite Borel measure μ, such that $\mu(X)=0$ if and only if X is of the first category in $\mathscr{X}$ (theorem (vii)).

1. Metric spaces. We base our consideration on the two following simple lemmas:

(i) *If a topological space $\mathscr{X}$ contains a dense set G_δ of measure μ zero, the decomposition* (*) *exists.*

For, if H is such a set G_δ, the set $K=\mathscr{X}-H$ is a set F_σ without interior points; thus K is of the first category in $\mathscr{X}$.

(ii) *If a topological space $\mathscr{X}$ is separable* [4]*, and if every enumerable subset of $\mathscr{X}$ is contained in a set G_δ of measure μ zero, then the decomposition* (*) *exists.*

For, if X is an enumerable dense subset of $\mathscr{X}$, and H is a set G_δ with $X\subset H$ and $\mu(H)=0$, then the set H satisfies the assumption of (i).

We now establish the theorems:

(iii) *The decomposition* (*) *exists for α-dimensional measures* [5]) *in separable metric spaces* (in particular for the Lebesgue measure in the n-dimensional Euclidean space [6]), for the linear measure in the plane, etc.).

In fact, an α-dimensional measure μ possesses the following property (δ) [7]) which implies the assumption of (ii):

(δ) Every Borel set X is contained in a set G_δ whose measure is equal to $\mu(X)$.

(iv) *Every σ-finite Borel measure μ in a metric space $\mathscr{X}$ possesses the property* (δ).

Suppose first $\mu(\mathscr{X})<\infty$. For every set $X\subset\mathscr{X}$ let $\nu_e(X)$ denote the lower bound of all numbers $\mu(G)$, where G is open and $G\subset X$. The so defined set function $\nu_e(X)$ is an outer measure in the sense of Carathéodory [8]). Let $\nu(X)$ denote the function $\nu_e(X)$ restricted to Borel sets $X\subset\mathscr{X}$. By a well-known theorem [9]), $\nu(X)$ is a Borel measure in $\mathscr{X}$. By the definition, $\mu(X)\leqslant\nu(X)$ for any Borel set $X\subset\mathscr{X}$. Consequently also $\mu(\mathscr{X}-X)\leqslant\nu(\mathscr{X}-X)$. Thus $\mu(\mathscr{X})=\mu(X)+\mu(\mathscr{X}-X)\leqslant\nu(X)+\nu(\mathscr{X}-X)=\nu(\mathscr{X})=\mu(\mathscr{X})<\infty$. Hence we have $\mu(X)=\nu(X)$ for every Borel set $X\subset\mathscr{X}$. Let $\{G_n\}$ be a sequence of open sets such that $X\subset G_n$ and $\mu(X)=\nu(X)\leqslant\mu(G_n)\leqslant\nu(X)+1/n$. The set $H=G_1G_2\ldots$ is a G_δ with $X\subset H$ and $\mu(X)=\mu(H)$, which proves the theorem in the case μ is finite.

Suppose now μ is a σ-finite measure. Let $X=X_1+X_2+\ldots$, $\mu(X_n)<\infty$, and let $X_nX_m=0$ for $n\neq m$. The set functions $\mu_n(X)=\mu(XX_n)$ for $n=1,2,\ldots$ are finite Borel measures in $\mathscr{X}$, and $\mu(X)=\mu_1(X)+\mu_2(X)+\ldots$ Let X be a Borel subset of $\mathscr{X}$, and let H_n be a set G_δ such that $X\subset H_n$ and $\mu_n(H_n)=\mu_n(X)$. The set $H_0=H_1H_2\ldots$ is a G_δ, $X\subset H_0$, and $\mu_n(X)=\mu_n(H_0)$ for $n=1,2,\ldots$ Thus $\mu(H_0)=\mu_1(H_0)+\mu_2(H_0)+\ldots=\mu_1(X)+\mu_2(X)+\ldots=\mu(X)$, q. e. d.

(v) *If $\mathscr{X}$ is a separable metric space, the decomposition* (*) *exists for each σ-finite Borel measure μ vanishing for all one-point sets* [10]).

This is an immediate consequence of (ii) and (iv).

We say that a cardinal number $\mathfrak{m}$ *has measure zero* if every finite measure, defined for all subsets of a set Y of potency $\mathfrak{m}$ and vanishing for all one-point sets, vanishes identically.

Ulam has proved that every cardinal number less than the first aleph inaccessible in the weak sense [11]) has measure zero [12]).

(vi) *If a metric space $\mathscr{X}$ contains a dense subset, the potency of which has measure zero, the decomposition* (*) *exists for any σ-finite measure μ vanishing for all one-point sets.*

Then there exists a decomposition [13]) $\mathscr{X}=N+S$, where $\mu(N)=0$ and S is separable.

[4]) i. e. contains an enumerable dense subset.

[5]) See Saks [5], pp. 53-54, and Hahn and Rosenthal [1], p. 106.

[6]) This result is known, but our proof is still simpler than other proofs.

[7]) See e. g. Hahn and Rosenthal [1], p. 106.

[8]) See Saks [5], p. 43.

[9]) See ibidem, p. 52.

[10]) Another proof of this theorem was given by Marczewski. See [3], p. 304.

[11]) An aleph $\mathfrak{p}=\aleph_\lambda>\aleph_0$ is *inaccessible in the weak sense* if λ is a limit number and if the condition $\mathfrak{p}_t<\mathfrak{p}$, where t runs over a set T of potency less than $\mathfrak{p}$, implies $\sum_{t\in T}\mathfrak{p}_t<\mathfrak{p}$. See Tarski [10], p. 69.

[12]) Ulam [11], p. 141 (Satz A).

[13]) See Marczewski and Sikorski [4], p. 137.

The set function $\nu(X)=\mu(XS)$ for $X\subset S$ is a σ-finite Borel measure in the separable metric space S. By (v) there exists a decomposition $S=H_0+K$, where $\nu(H_0)=0$ and K is of the first category in S.

Let $H=H_0+N$. The formula $\mathscr{X}=H+K$ gives the decomposition $(*)$ of $\mathscr{X}$.

In theorem (vi) the condition that μ be σ-finite is essential and cannot be replaced by the weaker property (δ). E. g. let $\mathscr{X}$ be a set of potency $\aleph_1$ with the metric $\varrho(x_1, x_2)=1$ for $x_1\neq x_2$, and let $\mu(X)=0$ if $\overline{\overline{X}}\leqslant\aleph_0$, and $\mu(X)=\infty$ if $\overline{\overline{X}}=\aleph_1$. The Borel measure μ has the property (δ) and $\mathscr{X}$ satisfies the condition given in (vi) since $\aleph_1$ has measure zero. However, the decomposition $(*)$ does not exist.

The condition that μ vanishes for every one-point set may be omitted if the space $\mathscr{X}$ is dense in itself or, more generally, if the set X_0 of all points $x\epsilon\mathscr{X}$ with $\mu((x))>0$ contains no isolated point. In fact, the formula $\nu(X)=\mu(X-X_0)$ defines a Borel measure in the space $\mathscr{X}-X_0$ with $\nu((x))=0$ for every $x\epsilon\mathscr{X}-X_0$. Thus there exists a decomposition $\mathscr{X}-X_0=H+K$ such that $\nu(H)=\mu(H)=0$ and K is of the first category. The measure μ being σ-finite, the set X_0 is enumerable, thus of the first category. The formula $\mathscr{X}=H+(K+X_0)$ establishes the decomposition $(*)$ of $\mathscr{X}$.

2. Applications to mappings. Lemma (ii) can be generalized as follows:

(vii) *Let μ be a Borel measure in a topological space $\mathscr{X}$, such that every enumerable set is contained in a set G_δ of measure μ zero. Then for every continuous mapping f of a separable topological space $\mathscr{X}_0$ into $\mathscr{X}$ there is a set $X_0\subset\mathscr{X}_0$, such that $\mu(f(X_0))=0$ and that the set $\mathscr{X}_0-X_0$ is of the first category in $\mathscr{X}_0$* [14].

Let D be an enumerable dense subset of $\mathscr{X}_0$. The set $f(D)$ being enumerable, there is a set $H\subset\mathscr{X}$ which is a G_δ, and $\mu(H)=0$. Let $X_0=f^{-1}(H)$. Obviously $\mu(f(X_0))=0$. Since $D\subset X_0$, the set X_0 is a G_δ dense in the space $\mathscr{X}_0$. Consequently, $\mathscr{X}_0-X_0$ is of the first category, q. e. d.

[14]) This theorem is known for the case of Lebesgue's measure, but its proof given for that case is more complicated than ours. See Sierpiński [7], p. 302, and Kuratowski [2], p. 311.

Evidently, the continuity of f may be replaced by the more general condition that f be continuous on a set $\mathscr{X}_0-K$, where K is of the first category in $\mathscr{X}_0$. In particular, theorem (vii) is true if f possesses the property of Baire [15]) and if $\mathscr{X}$ is a separable metric space.

The following theorem [16]) may be considered as a generalization of theorem (vi):

(viii) *Let ν be a σ-finite measure defined on a σ-additive field $\boldsymbol{F}$ of subsets of a set $\mathscr{X}_0$, and let f be a mapping of $\mathscr{X}_0$ into a metric space $\mathscr{X}$ dense in itself and containing a dense subset, the potency of which is of measure zero. If f is measurable (i. e. if $f^{-1}(X)\epsilon\boldsymbol{F}$ for every Borel set $X\subset\mathscr{X}$), then there is a set $X_0\epsilon\boldsymbol{F}$, such that $\nu(X_0)=0$ and that $f(\mathscr{X}_0-X_0)$ is of the first category.*

The formula $\mu(X)=\nu(f^{-1}(X))$ defines a σ-finite Borel measure in $\mathscr{X}$. By (vi) there is such a decomposition $\mathscr{X}=H+K$ that $\mu(H)=0$ and K is of the first category. Let $X_0=f^{-1}(H)$. Then $\nu(X_0)=\mu(H)=0$, and the set $f(\mathscr{X}_0-X_0)\subset K$ is of the first category.

3. Non-metrizable spaces. Let ν be a measure on a σ-additive field $\boldsymbol{F}$ of sets, and let $\boldsymbol{N}$ be the ideal of all sets of measure ν zero. The measure ν induces a measure $\bar{\nu}$ on the Boolean quotient algebra $\boldsymbol{F}/\boldsymbol{N}$; we set namely

$$\bar{\nu}(A)=\nu(X),$$

where $X\epsilon A\epsilon\boldsymbol{F}/\boldsymbol{N}$.

We say that a measure ν on $\boldsymbol{F}$ is *almost isomorphic* to a measure ν_1 on a σ-additive field $\boldsymbol{F}_1$ if there exists an isomorphism h of $\boldsymbol{F}/\boldsymbol{N}$ on $\boldsymbol{F}_1/\boldsymbol{N}_1$, such that,

$$\bar{\nu}_1(h(A))=\bar{\nu}(A) \quad \text{for every} \quad A\epsilon\boldsymbol{F}/\boldsymbol{N},$$

where $\boldsymbol{N}_1$ is the ideal of all sets of measure ν_1 zero, and $\bar{\nu}_1$ is the measure on $\boldsymbol{F}_1/\boldsymbol{N}_1$ induced by ν.

[15]) See Kuratowski [2], p. 306.

[16]) This theorem is known for Lebesgue's measure. See Sierpiński [6], p. 63.

(vii) *Let ν be a measure defined on a σ-additive field* $\boldsymbol{F}$. *Then there exists a normal, bicompact and totally disconnected* [18]) *space* $\mathscr{X}$, *and a Borel measure* μ *in* $\mathscr{X}$, *such that:*

(a) μ *is almost isomorphic to* ν;

(b) $\mu(X)=0$ *if and only if* X *is of the first category in* $\mathscr{X}$.

The Boolean algebra $\boldsymbol{F}/\boldsymbol{N}$ being complete [19]), $\boldsymbol{F}/\boldsymbol{N}$ is isomorphic to the quotient algebra $\boldsymbol{B}/\boldsymbol{K}$, where $\boldsymbol{B}$ is the field of all Borel subsets of Stone's space $\mathscr{X}$ constructed for the Boolean algebra $\boldsymbol{F}/\boldsymbol{N}$, and $\boldsymbol{K}$ is the ideal of all sets of the first category in $\mathscr{X}$ [20]). The space $\mathscr{X}$ is normal, bicompact and totally disconnected [21]). Let h be an isomorphism of $\boldsymbol{B}/\boldsymbol{K}$ on $\boldsymbol{F}/\boldsymbol{N}$, and let $\mu(X)=\bar{\nu}(h(A))$, where $X \epsilon \boldsymbol{B}$ and $X \epsilon A \epsilon \boldsymbol{B}/\boldsymbol{K}$. The Borel measure μ satisfies both the conditions (a) and (b).

It is to be remarked that $\mathscr{X}$ contains a dense subset of potency at most $\overline{\overline{\boldsymbol{F}}}$. If the Boolean algebra $\boldsymbol{F}/\boldsymbol{N}$ has no atom, the measure μ vanishes for every one-point set.

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[18]) A topological space $\mathscr{T}$ is *totally disconnected* if for every pair of its different points x,y there exists a both open and closed set $G \subset \mathscr{T}$ with $x \epsilon G$ and $y \epsilon \mathscr{T} - G$.

[19]) See e. g. Wecken [12], p. 380.

[20]) See Sikorski [8], p. 257.

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