ON A BANACH'S PROBLEM OF INFINITE MATRICES

BY

M. NOSARZEWSKA (WROCŁAW)

All matrices considered in this paper are infinite square matrices.

A matrix $B=(b_{k,l})$ is a *permutation* of a matrix $A=(a_{i,j})$ if there exists a one-to-one transformation $(k,l)=(k(i,j),l(i,j))$ of the set of all pairs of positive integers into itself, such that

$$a_{i,j}=b_{k(i,j),l(i,j)}.$$

If $k(i,j)=i$ for $i=1,2,\ldots$, i. e. if every element remains in its line, then B is said to be a *line permutation* of A. Analogously, if $l(i,j)=j$ for $j=1,2,\ldots$, i. e. if every element remains in its column, then B is said to be a *column permutation* of A.

S. Banach proposed the problem [1]) whether every permutation B of a matrix A is a superposition of a finite sequence of line permutations and of column permutations.

The purpose of this paper is to show that the answer to Banach's problem is affirmative:

Every permutation B of a matrix $A=(a_{i,j})$ is the result of five alternating line and column permutations.

Proof. Let $m=f(n)$ be a transformation of the set of all positive integers onto itself such that the set $f^{-1}(m)$ is infinite for every positive integer m.

First step: a line permutation. Let us order lexicographically the elements of A, i. e. let us put $a_{i,j}<a_{i',j'}$ if $i<i'$, or if $i=i'$ and $j<j'$. Let a_{i_1,j_1} be the first element of A which is in the $f(1)$-th line of B, and let $k_1=1+j_1$. By induction we define an infinite sequence $\{a_{i_n,j_n}\}$ of elements of A and an infinite increasing sequence $\{k_n\}$ of positive integers as follows:

$a_{i_{n+1},j_{n+1}}$ is the first element of A which is in the $f(n+1)$th line of B and which is different from the elements

1) Colloquium Mathematicum 1 (1948), p. 151, P32.

$$a_{i_1,j_1},\ldots,a_{i_n,j_n},a_{i_1,k_1},\ldots,a_{i_n,k_n}$$

(two elements of a matrix being considered as different if their pairs of indices are not the same);

k_{n+1} is the least integer greater than

$$j_1,\ldots,j_n,j_{n+1},k_1,\ldots,k_n.$$

It follows from this definition that

$$(i_n,j_n)\neq(i_m,k_m) \quad \text{for } n,m=1,2,\ldots,$$

$$\left.\begin{array}{l}(i_n,j_n)\neq(i_m,j_m)\\(i_n,k_n)\neq(i_m,k_m)\end{array}\right\} \text{for } n\neq m.$$

We now define a line permutation $A'=(a'_{i,j})$ of the matrix A by the formulae

$$a'_{i_n,j_n}=a_{i_n,k_n},\qquad a'_{i_n,k_n}=a_{i_n,j_n},$$

$$a'_{i,j}=a_{i,j}\quad \text{for all other pairs } (i,j).$$

Thus the element a'_{i_n,k_n} lies in the $f(n)$-th line of B.

Let $\{l_m\}$ be a sequence of positive integers defined by the equalities:

$$l_{k_n}=i_n,\quad\text{and}\quad l_m=1\quad\text{for all other } m.$$

The sequence $c=\{a'_{l_m,m}\}$ has the property that every column of A' contains exactly one element of c and, for every n, the sequence c contains an enumerable number of elements lying in the n-th line of B.

Second step: a column permutation. Let $c_n=\{a_{i_{n,m},j_{n,m}}\}_{m=1,2,\ldots}$ be the subsequence of c containing all elements of c which lie in the n-th line of B.

The permutation $A''=(a''_{i,j})$ of A' is defined by the equalities:

$$a''_{i_{n,m},j_{n,m}}=a'_{f(m),j_{n,m}},$$

$$a''_{f(m),j_{n,m}}=a'_{i_{n,m},j_{n,m}},$$

$$a''_{i,j}=a'_{i,j}\quad\text{for all other pairs } (i,j).$$

Every set $f^{-1}(s)$ being infinite, any line of A'' contains an enumerable number of elements from every line of B.

Let $\boldsymbol{d}_{s,r}=\{a''_{s,p_{s,r,n}}\}_{n=1,2,\ldots}$ denote the sequence containing all elements $a''_{s,p}$ from the s-th line of A'' which lie in the r-th line of B.

Third step: a line permutation. Let $\varpi(s,r)$ be an infinite square matrix of positive integers such that $\varpi(1,r)=r$ and that every positive integer appears exactly once in every line and exactly once in every column of this matrix. An example of such a matrix is the following one:

$\varpi(1,r)=r,\quad$ for $r=1,2,\ldots$

$$\varpi(2^q+1,r)=\begin{cases} r+2^q & \text{for } n2^{q+1}<r\leqslant n2^{q+1}+2^q \\ r-2^q & \text{for } n2^{q+1}+2^q<r\leqslant (n+1)2^{q+1} \end{cases}\quad \begin{matrix} q=0,1,2,\ldots \\ n=0,1,2,\ldots \end{matrix}$$

$\varpi(2^q+t,r)=\varpi(t,\varpi(2^q+1,r))$ for $1<t\leqslant 2^q$, $q=0,1,2,\ldots$, $r=1,2,\ldots$

Explicitly:

$$\begin{matrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 & 10 & \ldots \\
3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 & 11 & \ldots \\
4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 & 12 & \ldots \\
5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 13 & \ldots \\
6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 & 14 & \ldots \\
7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 & 15 & \ldots \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 16 & \ldots \\
9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 1 & \ldots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ldots
\end{matrix}$$

Let $A'''=(a'''_{i,k})$ be the matrix defined by the equality:

$$a'''_{s,p_{s,r,n}}=a''_{s,p_{s,\varpi(s,r),n}}.$$

In other words the construction of A''' is as follows: the first line of A''' is identical with that of A''; in the s-th line we put the elements of the sequence $\boldsymbol{d}_{s,\varpi(s,r)}$ under the elements of the sequence $\boldsymbol{d}_{1,r}$ in the same order. Obviously, the matrix A''' is a line permutation of A'' and it has the following property:

(i) every column of A''' contains exactly one element from the m-th line of B $(m=1,2,\ldots)$.

In fact, let us consider the p-th column of A'''. We have $p=p_{1,r,n}$ for some r and n. Thus

$$a'''_{s,p}=a'''_{s,p_{1,r,n}}=a''_{s,p_{s,\varpi(s,r),n}}.$$

By definition of $\varpi(s,r)$ there exists exactly one integer s such that $m=\varpi(s,r)$; and by definition of $\boldsymbol{d}_{s,r}$ the element $a'''_{s,p}$ lies in the m-th column of B if and only if $\varpi(s,r)=m$. This proves the property (i).

Fourth step: a column permutation. We put each element $a'''_{i,j}$ of A''' on the place (i,k), where k is the number of that line of B which contains the element $a'''_{i,j}$. By (i) we obtain in this way a matrix A'''' with the property:

(ii) every element lies in A'''' in the same line as in B.

Fifth step: a line permutation. We put each element $a''''_{i,k}$ of A'''' on the place (j,k), where j is the number of that column of B which contains the element $a''''_{i,k}$. By (ii) this line permutation gives the matrix B.

Remark. The theorem proved above holds also for matrices which have a finite number of lines and an infinite number of columns. The method of the proof is the same.