suffisante, mais aussi nécessaire pour l'existence du prolongement demandé, la difficulté que l'on rencontre si l'on veut (positivement) résoudre le problème **P 94** consiste à démontrer que la condition (C_σ) entraîne l'existence d'un module M satisfaisant à la condition $(C_\sigma\text{-}M)$.

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NOTE ON MINIMAX SOLUTIONS OF STATISTICAL DECISION PROBLEMS

BY

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Given two non-void sets A and B and let $r(x,y)$ be a function defined for every $x \epsilon A$ and $y \epsilon B$, such that

(1) $$r(x,y) \geqslant 0 \quad \text{for each} \quad x \epsilon A,\ y \epsilon B,$$

(2) $$\sup_{x \epsilon A} r(x,y) < \infty \quad \text{for each} \ y \epsilon B.$$

It is easy to show that there exists the least σ-algebra $\mathfrak{A}$ of subsets of A such that

(3) $$r(x,y) \text{ is } \mathfrak{A}\text{-measurable for each } y \epsilon B,$$

(4) $$\inf_{y \epsilon B} r(x,y) \text{ is } \mathfrak{A}\text{-measurable.}$$

Obviously, the condition (4) may be omitted if B is enumerable.

Let Ω_1 be the set of all probability measures in $\mathfrak{A}$. Then by (1) and (3)

(5) $$0 \leqslant f(\omega, y) = \int_A r(x,y)\, d\omega < \infty \quad \text{for each} \quad y \epsilon B,\ \omega \epsilon \Omega_1.$$

For each pair $\omega_1, \omega_2 \epsilon \Omega$ we shall define the function

(6) $$\varrho(\omega_1, \omega_2) = \sup_{X \epsilon \mathfrak{A}} [\omega_1(X) - \omega_2(X)].$$

It is easy to show that Ω_1 is a metric space with respect to the distance function (6). If $\Omega \subset \Omega_1$, we shall denote by $\bar{\Omega}$ the closure of Ω in Ω_1. A probability measure which assumes only the values 0 and 1 will be called an *elementary probability measure* and the set of all elementary probability measures will be denoted by Ω_0. If A contains at least two elements, then Ω_1 contains no isolated points, because, if $\omega \epsilon \Omega_1$, then there exists an $\omega' \epsilon \Omega_1$ distinct from ω, and we see at once that the sequence $\{(1-2^{-j})\,\omega(X) + 2^{-j}\omega'(X)\}_{j=1,2,3,\ldots}$ of pairwise distinct points of Ω_1 converges to $\omega(X)$. Obviously, Ω_0 is isolated and each two elements of Ω_0 have the distance 1. Throughout this paper we shall always suppose that A as well as B contains at least two elements, because this assumption excludes trivial cases.

The purpose of this paper is to show that the following two theorems hold:

Theorem 1. If $\Omega_0 \subset \bar{\Omega}$ and

$$\sup_{x\in A}\inf_{y\in B} r(x,y) = \inf_{y\in B}\sup_{x\in A} r(x,y) = c, \tag{7}$$

then

$$\sup_{\omega\in\Omega}\inf_{y\in B} f(\omega,y) = \inf_{y\in B}\sup_{\omega\in\Omega} f(\omega,y) = c. \tag{8}$$

Theorem 2. If $\Omega_0 \subset \Omega$ and

$$\max_{x\in A}\min_{y\in B} r(x,y) = \min_{y\in B}\max_{x\in A} r(x,y) = r(x_0,y_0), \tag{9}$$

then there exists an $\omega^*\in\Omega$ such that

$$\max_{\omega\in\Omega}\min_{y\in B} f(\omega,y) = \min_{y\in B}\max_{\omega\in\Omega} f(\omega,y) = f(\omega^*,y_0) = r(x_0,y_0). \tag{10}$$

The application of these two theorems to statistical decision problems is based on the following interpretations of A, B, and $r(x,y)$: According to Wald's general formulation of the statistical decision problem, A is a given set of probability measures defined for all Borel sets of the finite or infinite dimensional sample space, B is a given set of decision operations, i.e. a set of transformations of the sample space on the given set of decisions, and $r(x,y)$ is the risk function, i.e. the sum of expectations of a weight function and a cost function. This formulation may be found in Wald's paper [1] and [2]. The passage from $r(x,y)$ to $f(\omega,y)$ corresponds to the randomization of the probability measures x from A. On the other hand, the randomization of the decision operations seems unreasonable from the point of view of practical applications. Roughly speaking, the theorems 1 and 2 assert that the restriction to non-randomized decision operations is legitimate under very general conditions. In particular, this simple result may be used in the decision problem in [3].

The proof of Theorem 1 will result from the following three relations:

$$\sup_{\omega\in\Omega}\inf_{y\in B} f(\omega,y) \leqslant \inf_{y\in B}\sup_{\omega\in\Omega} f(\omega,y), \tag{11}$$

$$\inf_{y\in B}\sup_{\omega\in\Omega} f(\omega,y) = \inf_{y\in B}\sup_{x\in A} r(x,y), \tag{12}$$

$$\sup_{\omega\in\Omega}\inf_{y\in B} f(\omega,y) \geqslant \sup_{x\in A}\inf_{y\in B} r(x,y). \tag{13}$$

Proof of (11). Since

$$f(\omega,y) \leqslant \sup_{\omega\in\Omega} f(\omega,y) \quad \text{for each} \quad \omega\in\Omega,\ y\in B$$

we have

$$\inf_{y\in B} f(\omega,y) \leqslant \inf_{y\in B}\sup_{\omega\in\Omega} f(\omega,y) \quad \text{for each} \quad \omega\in\Omega$$

and the last inequality implies (11).

The proof of the inequality (11) under similar conditions may be found in Wald's paper [2] and it is reproduced here only for the sake of completeness.

Proof of (12). Obviously, it is sufficient to show that

$$\sup_{\omega\in\Omega} f(\omega,y) = \sup_{x\in A} r(x,y) \quad \text{for each} \quad y\in B. \tag{14}$$

Since

$$f(\omega,y) \leqslant \sup_{x\in A} r(x,y) \quad \text{for each} \quad \omega\in\Omega,\ y\in B,$$

we have

$$\sup_{\omega\in\Omega} f(\omega,y) \leqslant \sup_{x\in A} r(x,y) \quad \text{for each} \quad y\in B. \tag{15}$$

For each $y\in B$ and $\varepsilon>0$ let $X_y^{(\varepsilon)}$ be the set of all $x\in A$ for which

$$r(x,y) > \sup_{x\in A} r(x,y) - \varepsilon.$$

From the properties of supremum and using (2) and (3) we have $0 \neq X_y^{(\varepsilon)} \in \mathfrak{A}$ and the integral

$$\int_{X_y^{(\varepsilon)}} r(x,y)\,d\omega$$

exists for each $y\in B$, $\omega\in\Omega_1$ and $\varepsilon>0$. Furthermore, there exists an $\omega_y^{(\varepsilon)}\in\Omega_0$ such that

$$\omega_y^{(\varepsilon)}(X_y^{(\varepsilon)}) = 1 \quad \text{for each} \quad y\in B,\ \varepsilon>0,$$

because it is sufficient to choose an $x\in X_y^{(\varepsilon)}$ such that

$$\omega_y^{(\varepsilon)}(X) = \begin{cases} 1 & \text{if} \quad x\in X, \\ 0 & \text{if} \quad x\in X. \end{cases}$$

Since $\Omega_0 \subset \bar{\Omega}$, therefore, to each $\delta > 0$, $\varepsilon > 0$ and $y \in B$ there exists an $\omega_{y,\delta}^{(\varepsilon)} \in \Omega$ such that

(16) $$\omega_{y,\delta}^{(\varepsilon)}(X_y^{(\varepsilon)}) > 1 - \delta.$$

From the properties of supremum and using (16) we have, for each $y \in B$, $\varepsilon > 0$ and $\delta > 0$,

$$\sup_{\omega \in \Omega} f(\omega, y) \geqslant f(\omega_{y,\delta}^{(\varepsilon)}, y) = \int_A r(x,y)\, d\omega_{y,\delta}^{(\varepsilon)} \geqslant$$

$$\geqslant \int_{X_y^{(\varepsilon)}} r(x,y)\, d\omega_{y,\delta}^{(\varepsilon)} > (1-\delta)[\sup_{x \in A} r(x,y) - \varepsilon],$$

and therefore

$$\sup_{\omega \in \Omega} f(\omega, y) \geqslant \sup_{x \in A} r(x,y) \quad \text{for each} \quad y \in B.$$

Thus, using the opposite inequality (15) we obtain (14). The relation (12) is an immediate consequence of (14).

Proof of (13). We shall first show that

(17) $$\inf_{y \in B} f(\omega, y) \geqslant \int_A \inf_{y \in B} r(x,y)\, d\omega \quad \text{for each} \quad \omega \in \Omega.$$

Since

$$r(x,y) \geqslant \inf_{y \in B} r(x,y) \quad \text{for each} \quad x \in A,\ y \in B,$$

we have

$$f(\omega, y) \geqslant \int_A \inf_{y \in B} r(x,y)\, d\omega \quad \text{for each} \quad y \in B,\ \omega \in \Omega,$$

and (17) follows at once from this inequality. For the proof of (13) it is necessary to consider separately the following two cases:

(18) $$\sup_{x \in A} \inf_{y \in B} r(x,y) = 0,$$

(19) $$\sup_{x \in A} \inf_{y \in B} r(x,y) \neq 0.$$

We shall first suppose that (18) holds. Then by (17)

$$\sup_{\omega \in \Omega} \inf_{y \in B} f(\omega, y) \geqslant \sup_{\omega \in \Omega} \int_A \inf_{y \in B} r(x,y)\, d\omega = 0 = \sup_{x \in A} \inf_{y \in B} r(x,y),$$

i. e. (13) holds. Now let us suppose that (19) holds. Because of (2) and (11) we have

(20) $$0 < \sup_{x \in A} \inf_{y \in B} r(x,y) < \infty.$$

Before proving the inequality (13) under the condition (19) we shall first show that *if the sequence $\{\omega_j(X)\}_{j=1,2,3,\ldots}$ of probability measures converges to the probability measure $\omega(X)$ in the metric space Ω_1, then*

(21) $$\lim_{j \to \infty} \int_A \inf_{y \in B} r(x,y)\, d\omega_j = \int_A \inf_{y \in B} r(x,y)\, d\omega.$$

Let us denote by Γ the class of all decompositions $\mathfrak{C}$ of A in a finite number of pairwise disjoint sets from $\mathfrak{A}$ and put

$$g(\mathfrak{C}, X) = \inf_{x \in X \in \mathfrak{C}} \inf_{y \in B} r(x,y).$$

Then

$$\int_A \inf_{y \in B} r(x,y)\, d\omega = \sup_{\mathfrak{C} \in \Gamma} \sum_{X \in \mathfrak{C}} g(\mathfrak{C}, X)\, \omega(X),$$

$$\int_A \inf_{y \in B} r(x,y)\, d\omega_j = \sup_{\mathfrak{C} \in \Gamma} \sum_{X \in \mathfrak{C}} g(\mathfrak{C}, X)\, \omega_j(X) \quad \text{for} \quad j = 1,2,3,\ldots,$$

and we have

(22) $$\left| \int_A \inf_{y \in B} r(x,y)\, d\omega_j - \int_A \inf_{y \in B} r(x,y)\, d\omega \right| \leqslant \leqslant \sup_{\mathfrak{C} \in \Gamma} \sum_{X \in \mathfrak{C}} [g(\mathfrak{C}, X) | \omega_j(X) - \omega(X) |].$$

Since $\omega_j(X) - \omega(X)$ are σ-additive set functions in A, their absolute variations $\alpha_j(X)$ are measures in $\mathfrak{A}$ for $j = 1,2,3,\ldots$ and it is easy to show that, for $j = 1,2,3,\ldots$,

(23) $$|\omega_j(X) - \omega(X)| \leqslant \alpha_j(X) \leqslant 2 \sup_{Y \subset X,\ Y \in \mathfrak{A}} |\omega_j(Y) - \omega(Y)|.$$

Hence, because of (23), (22) and (6)

(24) $$\begin{aligned} &\left| \int_A \inf_{Y \in B} r(x,y)\, d\omega_j - \int_A \inf_{y \in B} r(x,y)\, d\omega \right| \leqslant \\ &\leqslant \int_A \inf_{y \in B} r(x,y)\, d\alpha_j \leqslant \alpha_j(A) \sup_{x \in A} \inf_{y \in B} r(x,y) \leqslant \\ &\leqslant 2 \sup_{Y \in \mathfrak{A}} |\omega_j(Y) - \omega(Y)| \sup_{x \in A} \inf_{y \in B} r(x,y) = \\ &= 2\varrho(\omega_j, \omega) \sup_{x \in A} \inf_{y \in B} r(x,y). \end{aligned}$$

Since by hypothesis the sequence $\{\omega_j(X)\}_{j=1,2,3,\ldots}$ converges to $\omega(X)$ in the metric space Ω_1 and (20) holds, therefore, for each $\varepsilon > 0$ there exists an index $m(\varepsilon)$ such that

(25) $$\varrho(\omega_j, \omega) < \varepsilon [2 \sup_{x \in A} \inf_{y \in B} r(x,y)]^{-1} \quad \text{for each index} \quad j > m(\varepsilon).$$

Because of (24) and (25) we have

$$\Big|\int_A \inf_{y\in B} r(x,y)\,d\omega_j - \int_A \inf_{y\in B} r(x,y)\,d\omega\Big| < \varepsilon \quad \text{for each index} \quad j < m(\varepsilon),$$

i. e. (21) holds. For each $x\in A$ let ω_x be the probability measure from Ω_0 for which

$$\omega_x(X) = \begin{cases} 1 & \text{if} \quad x\in X, \\ 0 & \text{if} \quad x \text{ non-}\in X. \end{cases} \tag{26}$$

Since by hypothesis $\Omega_0 \subset \overline{\Omega}$, there exists a sequence

$$\{\omega_x^{(j)}(X)\}_{j=1,2,3,\ldots}$$

of probability measures from Ω which converges to the probability measure $\omega_x(X)$. Obviously,

$$\sup_{\omega\in\Omega}\inf_{y\in B} f(\omega,y) \geqslant \inf_{y\in B} f(\omega_x^{(j)},y) \quad \text{for each} \quad x\in A,\ j=1,2,3,\ldots$$

and, because of (17), (21) and (26), we have, for each $x\in A$,

$$\sup_{\omega\in\Omega}\inf_{y\in B} f(\omega,y) \geqslant \lim_{j\to\infty}\inf_{y\in B} f(\omega_x^{(j)},y) \geqslant$$

$$\geqslant \lim_{j\to\infty}\int_A \inf_{y\in B} r(x,y)\,d\omega_x^{(j)} = \int_A \inf_{y\in B} r(x,y)\,d\omega_x = \inf_{y\in B} r(x,y).$$

The last relation gives at once (13) and Theorem 1 is thus proved.

The proof of Theorem 2 is very simple. By hypothesis $\Omega_0 \subset \Omega$; hence, by (26), we may substitute ω_{x_0} for ω^*. Then

$$f(\omega^*, x_0) = \int_A r(x, y_0)\,d\omega^* = r(x_0, y_0),$$

and Theorem 2 follows at once from Theorem 1.

Let us consider the following example: A consists of two elements 0 and 1, B is the closed interval $[0,2]$ and $r(0,y)=y$, $r(1,y)=2-y$. Then $\mathfrak{A}$ is the σ-algebra of all subsets of A. Furthermore let $\Omega=\Omega_1$. It may be seen at once that

$$\max_{x\in A}\min_{y\in B} r(x,y) = 0 < 1 = \min_{y\in B}\max_{x\in A} r(x,y),$$

$$\max_{\omega\in\Omega}\min_{y\in B} f(\omega,y) = 1 = \min_{y\in B}\max_{\omega\in\Omega} f(\omega,y).$$

This example shows that the converse of Theorem 1 and 2 does not hold.

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