ON A FUNCTIONAL EQUATION

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In his dendrometric investigations [1]) J. Perkal met the following question: for what functions is the mean value of the function in any interval attained at a point dividing the interval in the ratio α, where α is a given number, $0<\alpha<1$?

The aim of this paper is to solve this problem in a slightly generalized form, namely:

If a function, integrable in any sense, satisfies the functional equation

$$f(\alpha a+(1-\alpha)b)=\frac{k}{b-a}\int_a^b f(t)\,dt, \tag{1}$$

where α and k are constant, then $f(x)$ is linear.

Proof. As $f(x)=\text{constant}$ satisfies (1) with $k=1$ and any α, we may suppose further that

$$f(x)\neq\text{constant}. \tag{2}$$

Let a be fixed. As the right hand of (1) is a continuous function of $b\neq a$, the same is valid for the left side; a being arbitrary, $f(x)$ is continuous everywhere. Hence it follows that the right hand of (1) has a first derivative with respect to $b\neq a$. We get therefore by the same argument as above that $f(x)$ has a first derivative everywhere. Reasoning along the same line we see that $f(x)$ is infinitely differentiable and all the derivatives are continuous everywhere.

Multiplying (1) by $b-a$ and differentiating we get

$$(1-\alpha)(b-a)f'(\alpha a+(1-\alpha)b)+f(\alpha a+(1-\alpha)b)=kf(b). \tag{3}$$

Putting $b=a$ we get

$$f(a)=kf(a)$$

for arbitrary a.

[1]) Cf. this fascicle, p. 251.

By (2) we see that

$$k=1. \tag{4}$$

By (3), using $k=1$, we obtain further

$$(1-\alpha)f'(\alpha a+(1-\alpha)b)=\frac{f(b)-f(\alpha a+(1-\alpha)b)}{b-a}.$$

The difference between the arguments of the function in the numerator of the right hand being $\alpha(b-a)$, by the mean value theorem we get

$$(1-\alpha)f'(\alpha a+(1-\alpha)b)=\alpha f'(\xi),$$

where $a<\xi<b$. Let $b\to a$. Then

$$(1-\alpha)f'(a)=\alpha f'(a).$$

The function $f(x)$ not being constant, there are points a for which $f'(a)\neq 0$, and therefore

$$\alpha=1/2. \tag{5}$$

Using (4) and (5) the equation (3) takes the form

$$f(b)=f\left(\frac{a+b}{2}\right)+\frac{b-a}{2}f'\left(\frac{a+b}{2}\right).$$

The terms of the right hand of this equality are the first two terms of Taylor's formula, when expanding $f(x)$ around $\frac{a+b}{2}$. It follows that the quadratic remainder of Taylor's formula is zero, and therefore

$$f''(\xi)=0,\qquad \frac{a+b}{2}<\xi<b.$$

Now let $b\to a$. Then by the continuity of $f''(x)$ we get

$$f''(a)=0$$

for arbitrary a; therefore $f(x)$ is linear.

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