On parvient ainsi à l'égalité

$$\overline{G^*_{\iota_1}}\cdot\ldots\cdot\overline{G^*_{\iota_n}}=\overline{X}_{\iota_1}\cdot\ldots\cdot\overline{X}_{\iota_n}$$

qui, rapprochée de la double inclusion (29), montre que les ensembles G^*_ι satisfont au théorème B complété par (28).

2. Les suites *finies* $X_{\iota_1},\ldots,X_{\iota_n}$ dans (26) et (27) ne peuvent pas être remplacées par des suites *infinies*.

Pour s'en convaincre, on désignera par 1 l'ensemble des nombres réels et par E celui des nombres rationnels:

$$E=(r_1,r_2,\ldots).$$

Soit $X_n=(r_n)$. On a

$$X_1+X_2+\ldots=E,$$

tandis que

$$F(X_1)+F(X_2)+\ldots\neq 1.$$

Car la condition $E\cdot F(X_n)=X_n=(r_n)$ implique que l'ensemble fermé $F(X_n)$ est non-dense. La somme $F(X_1)+F(X_2)+\ldots$ est donc de I^e catégorie au sens de Baire.

ON THE SEPARABILITY OF TOPOLOGICAL SPACES

BY

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1. Consider the following six properties of a topological space [1] $\mathcal{X}$:

(B) *$\mathcal{X}$ possesses a basis* [2].

$(\overline{M})$ *Every transfinite strictly increasing sequence of open subsets of $\mathcal{X}$ is at most enumerable* [3].

$(\underline{M})$ *Every transfinite strictly decreasing sequence of open subsets of $\mathcal{X}$ is at most enumerable.*

(I) *Every isolated subset of $\mathcal{X}$ is at most enumerable.*

(D) *$\mathcal{X}$ contains an at most enumerable subset X which is dense in $\mathcal{X}$* [4].

(S) *Every class of mutually disjoint open subsets of $\mathcal{X}$ is at most enumerable.*

The following seven implications are true for any topological space [5]:

$$\text{(i)}\qquad \begin{array}{ccccc} (\overline{M}) & \rightarrow & (I) & \rightarrow & (S) \\ \uparrow & & \uparrow & & \uparrow \\ (B) & \rightarrow & (\underline{M}) & \rightarrow & (D) \end{array}$$

If the space is metric, the implication $(S)\rightarrow(B)$ is also true, i. e. all the properties (B), $(\overline{M})$, $(\underline{M})$, (I), (D), (S) are equivalent.

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[1] A space is called *topological* if it fulfils the three well-known axioms of Kuratowski. See C. Kuratowski, *Topologie I* (second edition), Monografie Matematyczne, Warszawa-Wrocław 1948, p. 20.

[2] I. e. an enumerable sequence of open sets such that every open subset of $\mathcal{X}$ is the sum of some subsequence of this sequence.

[3] The property $(\overline{M})$ is equivalent (for arbitrary topological spaces) to the following property: every class G of open sets contains an enumerable subclass G_0 such that $\Sigma(G)=\Sigma(G_0)$.

[4] I. e. $\overline{X}=\mathcal{X}$, where $\overline{X}$ denotes the *closure* of the set X.

[5] Cf. e. g. E. Marczewski, *Séparabilité et multiplication cartésienne des espaces topologiques*, Fundamenta Mathematicae 34 (1947), p. 127-143, see 1.2(i), 1.3(i) and (iii), p. 130-133. See also C. Kuratowski, op. cit., p. 146.

This equivalence does not hold for arbitrary topological spaces. In particular, five examples mentioned by Marczewski[6]) show that the implications $(\overline{M})\to(B)$, $(D)\to(\underline{M})$, $(I)\to(\underline{M})$, $(S)\to(D)$ and $(S)\to(I)$ are not true in general.

The purpose of this paper is to prove the following theorem:

(*) *The implications* (i) *and their logical consequences are the sole true* ***logical connexions*** *among the properties* (B), $(\overline{M})$, $(\underline{M})$, (I), (D), (S) *of topological spaces.*

2. Consider all the alternatives

(a) $$(P_1)+(P_2)+\ldots+(P_k),$$

where (P_i) is one of the properties (B), $(B)'$, $(\overline{M})$, $(\overline{M})'$, $(\underline{M})$, $(\underline{M})'$, (I), $(I)'$, (D), $(D)'$, (S), $(S)'$, $i=1,\ldots,k$, $k\leqslant 6$, and the sign $'$ denotes always the negation.

One can easily verify that every alternative (a) is either a logical consequence of the implications (i), or it implies, on account of (i), one of the following alternatives[7]):

(1) $(I)+(D)'$, (2) $(\overline{M})+(\underline{M})'$, (3) $(\overline{M})'+(D)$,

(4) $(B)+(\overline{M})'+(\underline{M})'$, (5) $(\overline{M})'+(\underline{M})+(D)'$, (6) $(I)+(D)+(S)'$,

(7) $(\overline{M})+(\underline{M})+(I)'+(D)'$, (8) $(B)'$, (9) (S).

It is well known that every sentence which is formed from the sentences (B), $(\overline{M})$, $(\underline{M})$, (I), (D), (S) and the logical signs of negation and implication is inferentially equivalent to a conjunction of a finite number of sentences of the form (a)[8]). Therefore, in order to prove (*) it is sufficient to show that the alternatives (1)-(9) are, in general, not true. This is evident for (8) and (9). Thus we must construct seven examples of topological spaces for which the following conjunctions are true respectively:

(1)′ $(I)'(D)$, (2)′ $(\overline{M})'(\underline{M})$, (3)′ $(\overline{M})(D)'$,

(4)′ $(B)'(\overline{M})(\underline{M})$, (5)′ $(\overline{M})(\underline{M})'(D)$, (6)′ $(I)'(D)'(S)$,

(7)′ $(\overline{M})'(\underline{M})'(I)(D)$.

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[6]) E. Marczewski, loco cit., p. 133.

[7]) This remark is due to A. Mostowski.

[8]) See e. g. A. Mostowski, *Logika Matematyczna*, Monografie Matematyczne, Warszawa-Wrocław 1948, p. 33.

3. If $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are two disjoint topological spaces, we define in the set $\mathcal{Y}_1+\mathcal{Y}_2$ the closure operation by the formula

$$\overline{Y}=\overset{1}{\overline{Y\cdot\mathcal{Y}_1}}+\overset{2}{\overline{Y\cdot\mathcal{Y}_2}} \quad \text{for} \quad Y\subset\mathcal{Y}_1+\mathcal{Y}_2,$$

where $\overset{i}{\overline{Y}}$ denotes the closure of a set $Y\subset\mathcal{Y}_i$ in the space $\mathcal{Y}_i$, $i=1, 2$. The topological space which we obtain in this way from $\mathcal{Y}_1+\mathcal{Y}_2$ will be denoted by $\mathcal{Y}_1\dotplus\mathcal{Y}_2$. The sets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are both open and closed in $\mathcal{Y}_1\dotplus\mathcal{Y}_2$. Therefore:

(ii) *The space* $\mathcal{Y}_1\dotplus\mathcal{Y}_2$ *possesses one of the properties* (B), $(\overline{M})$, $(\underline{M})$, (I), (D), (S) *if and only if both the spaces* $\mathcal{Y}_1$ *and* $\mathcal{Y}_2$ *possess this property.*

Let now $\mathcal{Y}$ be an arbitrary topological space and let E be an abstract enumerable set, $\mathcal{Y}\cdot E=0$. We define in $\mathcal{Y}+E$ the closure operation by the formula

$$\overline{Y}=\begin{cases}\overline{Y\cdot\mathcal{Y}}+Y\cdot E & \text{if the set } Y\cdot E \text{ is finite,}\\ \mathcal{Y}+E & \text{if the set } Y\cdot E \text{ is infinite,}\end{cases}$$

where $\overline{Y\cdot\mathcal{Y}}$ denotes obviously the closure of $Y\cdot\mathcal{Y}$ in the space $\mathcal{Y}$. The topological space which we obtain in this way from $\mathcal{Y}+E$ will be denoted by $D(\mathcal{Y})$.

(iii) *The space* $D(\mathcal{Y})$ *possesses the property* (D).

In fact, the set E is an enumerable dense subset of $D(\mathcal{Y})$.

(iv) *The space* $D(\mathcal{Y})$ *possesses the property* $(\overline{M})$ *or* $(\underline{M})$ *respectively if and only if the space* $\mathcal{Y}$ *possesses this property.*

In fact, the class of non-empty open subsets of $D(\mathcal{Y})$ is identical with the class of all sets G which can be represented in the form

$$G=G_0+(E-E_0),$$

where G_0 is an open subset of the space $\mathcal{Y}$ and E_0 is finite.

4. Now let us consider the four following topological spaces:

$$\mathcal{X}_0,\ \mathcal{X}_1,\ \mathcal{X}_2,\ \mathcal{X}_3.$$

$\mathcal{X}_0$ is the set of all ordinal number $\xi<\Omega$ (i. e. of first and second class). For every $\alpha\epsilon\mathcal{X}_0$ let Q_α denote the set of all $\xi\epsilon\mathcal{X}_0$ such that $\xi<\alpha$, and let $P_\alpha=\mathcal{X}_0-Q_\alpha$. If X is a finite subset of $\mathscr{X}_0$, obviously we set $\overline{X}=X$. If $X\subset\mathscr{X}_0$ is infinite, let α denote the least ordinal number such that $X\cdot Q_\alpha$ is infinite and let $\overline{X}=X+P_\alpha$. It is easy to prove that the so-defined closure $\overline{X}$ satisfies the three axioms of Kuratowski.

$\mathscr{X}_1$ is a non-enumerable set with the closure operation

$$\overline{X}=\begin{cases} X & \text{if } X \text{ is a finite set,}\\ \mathscr{X}_1 & \text{if } X \text{ is an infinite set.}\end{cases}$$

$\mathscr{X}_2$ is a non-enumerable set with the closure operation

$$\overline{X}=\begin{cases} X & \text{if } X \text{ is at most enumerable,}\\ \mathscr{X}_2 & \text{if } X \text{ is non-enumerable.}\end{cases}$$

$\mathscr{X}_3$ is a non-enumerable set with the closure operation

$$\overline{X}=X \quad \text{for every} \quad X\subset\mathscr{X}_3.$$

We may suppose that $\mathscr{X}_i\cdot\mathscr{X}_j=0$ and $\mathscr{X}_i\cdot D(\mathscr{X}_j)=0$ for $i\neq j$, $i,j=0,1,2,3$.

Then:

1° *The space $D(\mathscr{X}_3)$ possesses the property* (1)′.

In fact, $D(\mathscr{X}_3)$ possesses the property (D) by (iii) and it does not possess (I) since $\mathscr{X}_3$ is an isolated non-enumerable subset of $D(\mathscr{X}_3)$.

2° *The space $\mathscr{X}_0$ possesses the property* (2)′.

Since $\{Q_\alpha\}$ is a non-enumerable increasing sequence of open sets, the condition $(\overline{M})$ is not satisfied. The property $(\underline{M})$ follows from the fact that for every open set $G\subset\mathscr{X}_0$ either G is at most enumerable or $\mathscr{X}_0-G$ is finite.

3° *The space $\mathscr{X}_2$ possessess the property* (3)′.

The property $(\overline{M})$ follows from the fact that every non-empty open set is the complement of an at most enumerable set. Since every enumerable set is closed, $\mathscr{X}_2$ does not possess the property (D).

4° *The space $\mathscr{X}_1$ possesses the property* (4)′.

In fact, the class of all non-empty open subsets of $\mathscr{X}_1$ coincides with the class of all complements of finite subsets of $\mathscr{X}_1$. Therefore $\mathscr{X}_1$ possesses the properties $(\underline{M})$ and $(\overline{M})$.

Let $\{G_n\}$ be any sequence of non-empty open subsets of this space. $\mathscr{X}_1$ being non-enumerable, the set $G_1\cdot G_2\cdot\ldots$ is non-empty. Let $x\,\epsilon\,G_1\cdot G_2\cdot\ldots$ The open set $\mathscr{X}_1-(x)$ is not the sum of a subsequence of the sequence $\{G_n\}$. Thus $\mathscr{X}_1$ does not possess the property (B).

5° *The space $D(\mathscr{X}_2)$ possesses the property* (5)′.

$D(\mathscr{X}_2)$ possesses the properties $(\overline{M})$ and (D) by 3°, (iv) and (iii). Suppose that $D(\mathscr{X}_2)$ possesses the property $(\underline{M})$. By (iv) we obtain then that $\mathscr{X}_2$ possesses the property $(\underline{M})$, thus by (i) the property (D) also, in contradiction with 3°.

6° *The space $\mathscr{X}_3\dotplus D(\mathscr{X}_3)$ possesses the property* (6)′.

It possesses the property (S) by 3°, (i), (ii) and (iii). By 1°, 3° and (ii) it does not satisfy the conditions (I) and (D).

7° *The space $\mathscr{X}_0\dotplus D(\mathscr{X}_2)$ possesses the property* (7)′.

It possesses the properties (I) and (D) on account of (i), (ii), 2° and 5°. Since $\mathscr{X}_0$ does not possess the property $(\overline{M})$, and $D(\mathscr{X}_2)$ does not possess the property $(\underline{M})$ (see 2° and 5°), the space $\mathscr{X}_0\dotplus D(\mathscr{X}_2)$ does not possess these properties on account of (ii).

The theorem (*) is thus proved.

In order to prove only that the sole true *implications* among the properties (B), $(\overline{M})$, $(\underline{M})$, (I), (D), (S) are the implications (i) and their logical consequences, it is sufficient to show that there exist topological spaces with the properties (1)′, (2)′, (3)′ respectively. The examples $D(\mathscr{X}_3)$, $\mathscr{X}_0$, $\mathscr{X}_2$ are, I think, more elementary than the five examples given by Marczewski[9]).

5. The question arises whether the theorem (*) is true for normal[10]) spaces, i. e.:

P 51. Are the implications (i) and their logical consequences the sole true logical connexions among the properties (B), $(\overline{M})$, $(\underline{M})$, (I), (D) and (S) of *normal* topological spaces?

[9]) See footnote [6].

[10]) A space $\mathscr{X}$ is called *normal* if for any disjoint closed sets X_1, X_2 there exists an open set G such that $X_1\subset G$ and $\overline{G}\cdot X_2=0$.

This problem is unsolved. It is known only that

(v) *If $2^{\aleph_0} < 2^{\aleph_1}$, every topological completely normal*[11]) *space with the property* (D) *possesses also the property* (I).

Suppose that a completely normal space contains an enumerable dense subset X_0 and a non-enumerable isolated subset Y_0. For every set $Y \subset Y_0$ we have $\overline{Y}\cdot(Y_0 - Y) + Y\cdot\overline{(Y_0 - Y)} = 0$. Thus there exists an open set G_Y such that $Y \subset G_Y$ and $\overline{G}_Y\cdot(Y_0 - Y) = 0$. Let $X_Y = X_0 \cdot G_Y$. If $Y_1 \neq Y_2$, then $X_{Y_1} \neq X_{Y_2}$. The one-one mapping X_Y maps the class of all subsets of Y_0 in the class of all subsets of X_0 in contradiction with $2^{\aleph_0} < 2^{\aleph_1}$.

[11]) A space $\mathfrak{X}$ is called *completely normal* if for every two sets X_1, X_2 such that $\overline{X}_1\cdot X_2 + X_1\cdot\overline{X}_2 = 0$ there exists an open set G such that $X_1 \subset G$ and $\overline{G}\cdot X_2 = 0$. A space $\mathfrak{X}$ is completely normal if and only if every subspace $X \subset \mathfrak{X}$ is normal (see e. g. C. Kuratowski, op. cit., p. 130, Remarques).

REMARKS ON A PROBLEM OF BANACH

BY

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S. Banach has posed the following problem [1]):

When is it possible to define on a metric space X with a metric $\varrho(x_1, x_2)$ another metric $\varrho_1(x_1, x_2)$ such that

(1) *if* $\lim_{n=\infty} \varrho(x_n, x) = 0$, *then* $\lim_{n=\infty} \varrho_1(x_n, x) = 0$;

(2) *the metric space X_1 which we obtain from X by admitting the function $\varrho_1(x_1, x_2)$ as the metric is compact?*

It is easy to see that Banach's problem is equivalent to the question under what conditions a metric space X possesses the following property:

(B) *There exists a one-one continuous mapping f of X onto a compact metric space Y.*

It is clear that the geometrical image $\underset{xy}{E}[y = \varphi(x)]$ of an arbitrary real function $\varphi(x)$ $(0 \leqslant x \leqslant 1)$ possesses the property (B). The function f is then the projection on the x-axis.

W. Sierpiński has constructed a connected plain set S which is both F_σ and G_δ and which is the sum of an enumerable sequence $\{I_n\}$ of mutually disjoint simple arcs [2]). The set S does not possess the property (B). In fact, suppose that there exists a one-one continuous mapping f such that $f(S)$ is compact. Since S is connected, $f(S)$ would be a continuum. Since f is one-one, the continuum $f(S)$ would be the sum of the enumerable sequence $\{f(I_n)\}$ of mutually disjoint continuums, which is impossible [3]).

[1]) See this volume, p. 150, **P26**.

[2]) W. Sierpiński, *Sur quelques propriétés topologiques du plan*, Fundamenta Mathematicae 4 (1923), p. 5. I_n is the sum of the segment $x = 1/n$, $0 \leqslant y \leqslant 1$ and of the part of the circle $x^2 + y^2 = 1/n^2$, where either $x \leqslant 0$ or $y \leqslant 0$.

[3]) See W. Sierpiński, Tôhoku Mathematical Journal 13 (1918), p. 300, and F. Hausdorff, *Mengenlehre*, Berlin-Leipzig 1927, p. 162.