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Observability of small solutions of linear differential-algebraic systems with delays

by

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Abstract: In this paper we investigate observability of small solutions of linear differential-algebraic systems with delays (DAD), i.e. solutions that vanish after some finite time. In particular, we prove existence of two kinds of small solutions of DAD systems. The main result are the rank type conditions for observability of three kinds of small solutions of linear differential-algebraic systems with delay.

Keywords: observability, small solutions, differential-algebraic systems, time-delay.

1. Introduction

The behaviour of a number of control systems (e.g. air traffic control, chemical engineering, transportation, manufacturing systems, robotics) can be modelled by differential and algebraic equations with delay. They are models of inhomogeneous systems (Grossman et al., 1993; de la Sen, 1996). These systems are described by a combination of differential and difference equations, with some variables being continuous and some other piecewise continuous. In that sense they are hybrid systems. It should be noted that the term "hybrid systems" has been widely used in the literature in various senses (Grossman et al., 1993; de la Sen, 1996; Marchenko, Poddubnaya, 2002). Hybridity, however, reflects a double structure of the systems and so linear differential-algebraic systems with delays belong to the "hybrid" class.

Observability of linear differential-algebraic systems with delays (DAD systems) has been studied for some years (Marchenko, Poddubnaya, Zaczkiewicz, 2006; Marchenko, Zaczkiewicz, 2005), but there are still open questions concerning the classification of observability of DAD systems. The aim of this paper is to establish observability of nontrivial small solutions of DAD systems, which is missing in the current classification.

A small solution of a DAD system is a solution that goes to zero faster than any exponential function. Existence of such solutions for linear retarded systems was proved by Henry (1970) and later by Kappel (1976) for linear neutral systems. Lunel (1986) gave an explicit characterization of the smallest possible time for which small solutions vanish. Observability of small solutions for the retarded case was first studied by Manitius (1982) and for general neutral systems with output delays by Salamon (1984).

2. Preliminaries

In this paper, we investigate the simplest linear time invariant differential-algebraic systems with delays (DAD):

$$\dot{x}(t) = A_{11}x(t) + A_{12}y(t),\ t > 0, \tag{1a}$$

$$y(t) = A_{21}x(t) + A_{22}y(t-h),\ t \geq 0, \tag{1b}$$

with output

$$z(t) = B_1x(t) + B_2y(t), \tag{1c}$$

where $x(t) \in \mathbb{R}^n,\ y(t) \in \mathbb{R}^m,\ z(t) \in \mathbb{R}^r, t \geq 0; A_{11} \in \mathbb{R}^{n\times n},\ A_{12} \in \mathbb{R}^{n\times m},\ A_{21} \in \mathbb{R}^{m\times n},\ A_{22} \in \mathbb{R}^{m\times m},\ B_1 \in \mathbb{R}^{r\times n},\ B_2 \in \mathbb{R}^{r\times m}$ are constant (real) matrices, $0 < h$ is a constant delay. We regard an absolutely continuous n-vector function $x(\cdot)$ and a piecewise continuous m-vector function $y(\cdot)$ as a solution of System (1) if they satisfy the equation (1a) for almost everywhere $t > 0$ and (1b) for $t \geq 0$.

System (1) should be completed with initial conditions:

$$x(+0) = x_0,\ y(\tau) = \psi(\tau),\ \tau \in [-h, 0), \tag{2}$$

where $x_0 \in \mathbb{R}^n;\ \psi \in PC([-h,0),\mathbb{R}^m)$ and $PC([-h,0),\mathbb{R}^m)$ denotes the set of piecewise continuous m-vector-functions on $[-h, 0]$. Observe that $y(t)$ at $t = 0$ is determined from the equation (1b).

3. Small solutions

In this section, following retarded functional-differential case described by Hale and Lunel (1993), we introduce the concept of small solutions for DAD systems.

Let $E(h)$ denote the exponential type of $h : \mathbb{C} \to \mathbb{C}$, assuming h is an entire function of order 1, for details see Boas (1954). Then

$$E(h) = \limsup_{|s|\to\infty} \frac{\log |h(s)|}{|s|}.$$

For $h : \mathbb{C} \to \mathbb{C}^q$, the exponential type of h is defined by

$$E(h) = \max_{1 \le j \le q} E(h_j), \text{ where } h = cal\{h_1, \dots, h_q\}.$$

Let $\Delta(p)$ be the characteristic matrix function

$$\Delta(p) = \begin{pmatrix} pI_n - A_{11} & -A_{12} \\ -A_{21} & I_m - A_{22}e^{-ph} \end{pmatrix}.$$

The matrix function $\Delta(p)$ appears by applying the Laplace transform to System (1). Let $\det \Delta(p)$ be the determinant of $\Delta(p)$. It follows from the above that the exponential type of $\det \Delta(p)$ is less than or equal mh. Define ε by

$$E(\det \Delta(p)) = mh - \varepsilon.$$

Let $\operatorname{adj} \Delta(p)$ be the matrix function of cofactors of $\Delta(p)$. Since the cofactors C_{ij} are $(n + m - 1) \times (n + m - 1)$ subdeterminants of $\Delta(p)$, the exponential type of the cofactors is less than or equal to mh. Define σ by

$$\max_{1 \le i,j \le n+m} E(C_{ij}(p)) = mh - \sigma.$$

We present some useful tools for computing the exponential type of functions.

Lemma 1 (Lunel, 1993) *Let F and G be entire functions such that F and G are $O(z^l)$, $l \in \mathbb{Z}$, in the closed right half plane. Then*

$$E(F \cdot G) = E(F) + E(G).$$

We can now prove the following result.

Theorem 1 *For $x(\cdot)$, $y(\cdot)$ being solutions of System (1) the following implications hold: i) if*

$$\forall k \in \mathbb{Z}\ x(t)e^{kt} \to 0 \text{ as } t \to +\infty, \tag{3a}$$

then $x(t) = 0$ for all $t \ge \varepsilon - \sigma$;
ii) if

$$\forall k \in \mathbb{Z}\ y(t)e^{kt} \to 0 \text{ as } t \to +\infty, \tag{3b}$$

then $y(t) = 0$ for all $t \ge \varepsilon - \sigma$.

Proof. Let $\hat{x}(s) = \int_0^\infty e^{-st}x(t)dt$ and $\hat{y}(s) = \int_0^\infty e^{-st}y(t)dt$. By (3), $\hat{x}(s)$ in case *i*) (and $\hat{y}(s)$ in case *ii*) is an entire function of order 1. Laplace transform of System (1) yields

$$\Delta(s) \cdot \begin{pmatrix} \hat{x}(s) \\ \hat{y}(s) \end{pmatrix} = \begin{pmatrix} x_0 \\ A_{22}e^{-sh} \int_{-h}^0 e^{-\tau s}\psi(\tau)d\tau \end{pmatrix}.$$

Hence

$$\det \Delta(s) \cdot \begin{pmatrix} \hat{x}(s) \\ \hat{y}(s) \end{pmatrix} = \text{adj } \Delta(s) \cdot \begin{pmatrix} x_0 \\ A_{22}e^{-sh} \int_{-h}^{0} e^{-\tau s}\psi(\tau)d\tau \end{pmatrix}. \tag{4}$$

To compute the exponential growth of the right-hand side let us write it as follows

$$\text{adj } \Delta(s) \cdot \begin{pmatrix} x_0 \\ A_{22}e^{-sh} \int_{-h}^{0} e^{-\tau s}\psi(\tau)d\tau \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \cdot \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \tag{5}$$

where $K_{11} \in \mathbb{C}^{n\times n}$, $K_{12} \in \mathbb{C}^{n\times m}$, $K_{21} \in \mathbb{C}^{m\times n}$, $K_{22} \in \mathbb{C}^{m\times m}$, $K_1 \in \mathbb{R}^{n\times 1}$, $K_2 \in \mathbb{C}^{m\times 1}$. We have (for details see Hale and Lunel, 1993)

$$\begin{aligned}
&E(K_{11}) = mh - \sigma,\ E(K_{12}) = (m-1)h - \sigma,\\
&E(K_{21}) = (m-1)h - \sigma,\\
&E(K_{22}) = (m-1)h - \sigma,\\
&E(K_1) = 0,\\
&E(K_2) = h.
\end{aligned}$$

By the representation above, taking into account (4), (5) and Lemma 1, we compute the exponential type of $\hat{x}(s)$ and $\hat{y}(s)$ as follows

$$\begin{aligned}
&E(\hat{x}(s)) = E((K_{11}\cdot K_1 + K_{12}\cdot K_2)/\det\Delta(s)),\\
&E(K_{11}\cdot K_1) = mh - \sigma + 0,\\
&E(K_{12}\cdot K_2) = (m-1)h - \sigma + h,\\
&E(\hat{x}(s)) \le mh - \sigma - (mh - \varepsilon) = \varepsilon - \sigma.
\end{aligned}$$

Similarly

$$\begin{aligned}
&E(\hat{y}(s)) = E((K_{21}\cdot K_1 + K_{22}\cdot K_2)/\det\Delta(s)),\\
&E(K_{21}\cdot K_1) = (m-1)h - \sigma + 0,\\
&E(K_{22}\cdot K_2) = (m-1)h - \sigma + h,\\
&E(\hat{y}(s)) \le mh - \sigma - (mh - \varepsilon) = \varepsilon - \sigma.
\end{aligned}$$

Hence, by the Paley-Wiener theorem, $x(t) = 0$ and $y(t) = 0$ for all $t \ge \varepsilon - \sigma$. This proves the theorem. ∎

Now, we can define the following notions.

A solution of System (1) is said to be trivial if it vanishes for $t \ge 0$.

Definition 1 *We say that System (1) has a nontrivial small solution if there exists a solution $x(\cdot)$, $y(\cdot)$ such that conditions (3) hold and at least $x(\cdot)$ or $y(\cdot)$ is not trivial.*

Definition 2 *We say that System (1) has a nontrivial small solution with respect to x if there exists a solution $x(\cdot)$, $y(\cdot)$ such that condition (3a) holds and $x(\cdot)$ is not trivial.*

Similarly

Definition 3 *We say that System (1) has a nontrivial small solution with respect to y if there exists a solution $x(\cdot)$, $y(\cdot)$ such that condition (3b) holds and $y(\cdot)$ is not trivial.*

We illustrate the notions introduced by examples.

Example 1 Consider System (1) of the form :

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 1 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 1 \end{pmatrix} y(t), \\ y(t) &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} x(t) + \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} y(t-h), \end{aligned} \tag{6}$$

with initial conditions

$$x(+0) = -h + h^2, \qquad y(\tau) = \begin{pmatrix} 3+4\tau \\ 1+\tau \end{pmatrix}, \ \tau \in [-h, 0).$$

The characteristic matrix is given by

$$\Delta(s) = \begin{pmatrix} s-1 & -1 & 1 \\ -2 & 1+2e^{-sh} & -4e^{-sh} \\ -1 & e^{-sh} & 1-2e^{-sh} \end{pmatrix} \text{ with determinant } \det\Delta(s) = s-2.$$

Then $E(\det\Delta(p)) = 0 = 2h - 2h$, so $\varepsilon = 2h$. The cofactor

$$C_{12}(s) = -\begin{vmatrix} s-1 & 1 \\ -1 & 1-2e^{-sh} \end{vmatrix} = 2se^{-sh} - 2e^{-sh} - s$$

has exponential type h, then $\sigma = h$ and $\varepsilon - \sigma$=h.

Upon computing the solutions, we obtain

$$\begin{aligned} \overline{x}(t) &= ((t-h)+1)(t-h) \\ \overline{y}(t) &= \begin{pmatrix} 2((t-h)^2 - 1 - (t-h)) \\ (t-h)^2 - 1 - (t-h) \end{pmatrix} \text{ for } t \geq 0. \end{aligned}$$

It is easy to check that $\overline{x}$ and $\overline{y}$ satisfy (6) and there exist initial conditions for which $\overline{x}$, $\overline{y}$ vanish for all $t \geq h$, then such a system has a nontrivial small solution.

Example 2 Let us take System (1) of the form:

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \end{pmatrix} y(t), \\ y(t) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y(t-h), \end{aligned} \tag{7}$$

with initial condition

$$x(+0) = 1, \qquad\qquad y(\tau) = \begin{pmatrix} \tau \\ \tau \end{pmatrix}, \; \tau \in [-h, 0).$$

By integrating, we obtain

$$\begin{aligned} x_1(t) &= (2-h)e^t + h - t - 1, \\ y_1(t) &= \begin{pmatrix} t-h \\ 0 \end{pmatrix} \text{ for } t \in [0, h) \end{aligned}$$

and

$$\begin{aligned} x_2(t) &= (2-h)e^t - e^{t-h}, \\ y_2(t) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for } t \geq h. \end{aligned}$$

So we obtain a nontrivial small solution of the system with respect to y.

4. Observability of small solutions

Definition 4 (observability of nontrivial small solutions) *The nontrivial small solutions of System (1) are said to be observable if every nontrivial small solution has a nonzero output for some $t \geq 0$. This means that*

$$\exists T > 0 \;\; \left.\begin{aligned} x(t) &= 0 \;\forall\, t \geq T \\ y(t) &= 0 \;\forall\, t \geq T \\ z(t) &= 0 \;\forall\, t \geq \,0 \end{aligned}\right\} \Rightarrow x(t) = 0,\; y(t) = 0,\; \forall t \geq 0.$$

Theorem 2 *The nontrivial small solutions of System (1) are observable if and only if the following conditions hold:*

$$i)\; \max_{\lambda \in \mathbb{C}} \operatorname{rank} \begin{bmatrix} A_{11} - \lambda I_n & A_{12} & 0 \\ A_{21} & -I_m & A_{22} \\ 0 & A_{22} & 0 \\ B_1 & B_2 & 0 \end{bmatrix} = n + m + \operatorname{rank} A_{22}, \tag{8}$$

$$ii)\; \operatorname{rank} \begin{bmatrix} B_2 \\ B_2 A_{22} \\ \vdots \\ B_2 (A_{22})^{m-1} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B_2 \\ B_2 A_{22} \\ \vdots \\ B_2 (A_{22})^{m-1} \\ A_{22} \end{bmatrix}. \tag{9}$$

Proof. Introduce notation:

$$A(\lambda) = \begin{bmatrix} A_{11} - \lambda I_n & A_{12} & 0 \\ A_{21} & -I_m & A_{22} \\ 0 & A_{22} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & B_2 & 0 \end{bmatrix}, \tag{10}$$

$$K = \max_{\lambda\in\mathbb{C}} \operatorname{rank} \begin{bmatrix} A(\lambda) \\ B \end{bmatrix}, \ k = \operatorname{rank} A_{22}.$$

Then K is always less than or equal to $n+m+k$.

Necessity

Suppose that $K < n+m+k$. We shall prove that there exists a nontrivial small solution of (1) with zero output in three steps.

Step 1. There exist polynomials

$$l(\lambda)=\sum_{j=0}^{\nu} l_j\lambda^j, \ p(\lambda)=\sum_{j=0}^{\nu} p_j\lambda^j, \ q(\lambda)=\sum_{j=0}^{\nu} q_j\lambda^j,$$

with $l \in \mathbb{R}^n[\lambda], \ p \in \mathbb{R}^m[\lambda], \ q \in \mathbb{R}^m[\lambda]$ such that $l(\lambda) \not\equiv 0$ or $p(\lambda) \not\equiv 0$ and

$$A(\lambda)\begin{pmatrix} l(\lambda) \\ p(\lambda) \\ q(\lambda)\end{pmatrix} = 0, \ B\begin{pmatrix} l(\lambda) \\ p(\lambda) \\ q(\lambda)\end{pmatrix} = 0 \,\forall\lambda \in \mathbb{C}. \tag{11}$$

Proof. Let $M(\lambda)$ and $N(\lambda)$ be unimodular matrices of appropriate size such that

$$M(\lambda)\begin{bmatrix} A(\lambda) \\ B \end{bmatrix} N(\lambda) = \begin{bmatrix} \alpha_1(\lambda) & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & \alpha_K(\lambda) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

is in Smith-form and

$$\begin{bmatrix} A(\lambda) \\ B \end{bmatrix} N(\lambda) = \begin{bmatrix} \beta_{11}(\lambda) & \cdots & \beta_{K1}(\lambda) & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & & \vdots \\ \beta_{1\ 2m+n+r}(\lambda) & & \beta_{K\ 2m+n+r}(\lambda) & 0 & \cdots & 0 \end{bmatrix}.$$

Then the last $2m+n-K$ columns $\begin{pmatrix} l^j(\lambda) \\ p^j(\lambda) \\ q^j(\lambda)\end{pmatrix}$, $j = K+1, \ldots, n+2m$, of $N(\lambda)$ satisfy (11). Now, suppose that the polynomials $l^j(\lambda)$ and $p^j(\lambda)$ vanish identically. The determinant of $N(\lambda)$ equals ± 1, then the $q^j(\lambda)$ are linearly independent (for every $\lambda \in \mathbb{C}$) and satisfy $A_{22}q^j(\lambda) = 0$. This implies that

$$\operatorname{rank} A_{22} \leq m - (2m+n-K) = K - n - m < k,$$

which is a contradiction. ■

Let us define $l_j = 0,\ p_j = q_j = 0$ for $j \in \mathbb{Z}$ and $j \notin \{0, \dots, \nu\}$.

Step 2. The following equations hold for all $j \in \mathbb{Z}$

$$\begin{aligned}
0 &= A_{11}l_j - l_{j-1} + A_{12}p_j,\\
0 &= A_{21}l_j - p_j + A_{22}q_j,\\
0 &= A_{22}p_j,\\
0 &= B_1 l_j + B_2 p_j.
\end{aligned} \tag{12}$$

Proof. These equations follow from (11) by the comparison of coefficients. For all $\lambda \in \mathbb{C}$ we have

$$0 = [A_{11} - \lambda I_n]l(\lambda) + A_{12}p(\lambda) = \sum_{j=0}^{\nu}(A_{11}l_j + A_{12}p_j)\lambda^j - \sum_{j=0}^{\nu} l_j \lambda^{j+1}$$

$$= \sum_{j=0}^{\nu+1}(A_{11}l_j - l_{j-1} + A_{12}p_j)\lambda^j,$$

$$0 = A_{21}l(\lambda) - p(\lambda) + A_{22}q(\lambda) = \sum_{j=0}^{\nu}(A_{21}l_j - p_j + A_{22}q_j)\lambda^j,$$

$$0 = A_{22}p(\lambda) = \sum_{j=0}^{\nu} A_{22}p_j\lambda^j,$$

$$0 = B_1 l(\lambda) + B_2 p(\lambda) = \sum_{j=0}^{\nu}(B_1 l_j + B_2 p_j)\lambda^j.$$

This proves (12). ■

Step 3.The function

$$x(t) = \begin{cases} \sum_{j=0}^{\nu} l_{\nu-j}\frac{(t-h)^{j+1}}{(j+1)!}, & 0 \ \le\ t < h,\\ 0, & h \ \le\ t < \infty,\end{cases}$$

$$y(t) = \begin{cases} \sum_{j=0}^{\nu} q_{\nu-j}\frac{(t)^{j+1}}{(j+1)!}, & -h \ \le\ t < 0,\\ \sum_{j=0}^{\nu} p_{\nu-j}\frac{(t-h)^{j+1}}{(j+1)!}, & 0 \ \le\ t < h,\\ 0, & h \ \le\ t < \infty,\end{cases}$$

defines a nontrivial small solution of System (1) with zero output.

Proof. First note that $x(t)$ or $y(t)$ do not vanish identically for $0 \le t < h$ since $l(\lambda)$ or $p(\lambda)$ is a nonzero polynomial. Secondly, it is easy to see that $x(t)$ or $y(t)$ is absolutely continuous for $t \ge 0$ or $t \ge -h$. Finally, it can be proved — by the use of (12) — that $x(t)$ and $y(t)$ satisfy System (1) for almost every $t \ge 0$ and that the output $z(t)$ given by (1c) vanishes for $t \geq 0$. We have the following:

$$\dot{x}(t) = \sum_{j=0}^{\nu} l_{\nu-j}\frac{(t-h)^j}{j!}$$
$$= \sum_{j=0}^{\nu}(A_{11}l_{\nu-j+1} + A_{12}p_{\nu-j+1})\frac{(t-h)^j}{j!} + (A_{11}l_0 + A_{12}p_0)\frac{(t-h)^{\nu+1}}{(\nu+1)!}$$
$$= A_{11}\sum_{j=0}^{\nu} l_{\nu-j}\frac{(t-h)^{j+1}}{(j+1)!} + A_{12}\sum_{j=0}^{\nu} p_{\nu-j}\frac{(t-h)^{j+1}}{(j+1)!} = A_{11}x(t) + A_{12}y(t),$$
$$y(t) = \sum_{j=0}^{\nu} p_{\nu-j}\frac{(t-h)^{j+1}}{(j+1)!} = \sum_{j=0}^{\nu}(A_{21}l_{\nu-j} + A_{22}q_{\nu-j})\frac{(t-h)^{j+1}}{(j+1)!}$$
$$= A_{21}x(t) + A_{22}y(t-h),$$

for $\tau \geq 2h$: $y(\tau) = y(t+h) = A_{21}x(t+h) + A_{22}y(t) = A_{22}\sum_{j=0}^{\nu} p_{\nu-j}\frac{(t-h)^{j+1}}{(j+1)!}$

$$= \sum_{j=0}^{\nu} A_{22}p_{\nu-j}\frac{(t-h)^{j+1}}{(j+1)!} = 0,$$
$$z(t) = B_1x(t) + B_2y(t) = B_1\sum_{j=0}^{\nu} l_{\nu-j}\frac{(t-h)^{j+1}}{(j+1)!} + B_2\sum_{j=0}^{\nu} p_{\nu-j}\frac{(t-h)^{j+1}}{(j+1)!} =$$
$$= \sum_{j=0}^{\nu}(B_1l_{\nu-j} + B_2p_{\nu-j})\frac{(t-h)^{j+1}}{(j+1)!} = 0. \qquad \blacksquare$$

Sufficiency

This part of the proof falls naturally into two parts: continuous and discontinuous. The first part considers the continuous part of a solution $y(t)$. The discontinuous part is discrete and finite because by (2) $y(t) \in PC([ih,(i+1)h),\mathbb{R}^m)$, $i = -1,,0,1,\ldots$ and it has a finite number of discontinuities on $[-h,0)$, they take place at points $T_\alpha = \{t_1,\ldots,t_\alpha\}$.

Continuous Part

Suppose that $K = n+m+k$ and let $x(t), y(t)$, $t \geq -h$, be a solution of (1) such that $x(t) = 0$, $y(t) = 0$ for $t \geq h$ and $z(t) = 0$ for $t \geq 0$. Then we prove in three steps that $x(t) = 0$ and $y(t) = 0$ for $t \geq 0$.

Step 1. The complex functions

$$\hat{x}(\lambda) = \int_0^h e^{-\lambda t}x(t)dt, \ \hat{y}(\lambda) = \int_0^h e^{-\lambda t}y(t)dt,$$
$$\hat{\hat{y}}(\lambda) = \int_0^{2h} e^{-\lambda t}y(t-h)dt,$$

satisfy the equation

$$\begin{bmatrix} A_{11}-\lambda I_n & A_{12} & 0 \\ A_{21} & -I_m & A_{22} \\ 0 & A_{22} & 0 \\ B_1 & B_2 & 0 \end{bmatrix} \begin{pmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \\ \hat{\hat{y}}(\lambda) \end{pmatrix} = \begin{bmatrix} -x(0) \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{13}$$

Proof. For every $\lambda \in \mathbb{C}$ we have

$$[A_{11}-\lambda I_n]\hat{x}(\lambda) + A_{12}\hat{y}(\lambda) = \int_0^h e^{-\lambda t} A_{11} x(t)dt - \int_0^h \lambda e^{-\lambda t} x(t)dt$$

$$+ \int_0^h e^{-\lambda t} A_{12} y(t)dt =$$

$$= \int_0^h e^{-\lambda t}(A_{11}x(t) - \dot{x}(t) + A_{12}y(t))dt - x(0) = -x(0),$$

$$A_{21}\hat{x}(\lambda) - \hat{y}(\lambda) + A_{22}\hat{\hat{y}}(\lambda) = \int_0^h e^{-\lambda t} A_{21} x(t)dt - \int_0^h e^{-\lambda t} y(t)dt +$$

$$\int_0^{2h} e^{-\lambda t} A_{22} y(t-h)dt =$$

$$= \int_0^{2h} e^{-\lambda t}(A_{21}x(t) - y(t) + A_{22}y(t-h))dt = 0,$$

$$A_{22}\hat{y}(\lambda) = \int_0^h e^{-\lambda t} A_{22} y(t)dt = 0,$$

$$B_1\hat{x}(\lambda) + B_2\hat{y}(\lambda) = \int_0^h e^{-\lambda t}(B_1x(t) + B_2y(t))dt = \int_0^h e^{-\lambda t} z(t)dt = 0. \qquad \blacksquare$$

Step 2.

There exist matrices $D \in \mathbb{R}^{(n+m+k)\times(n+m+k+r)}[\lambda]$ and $R \in \mathbb{R}^{k\times m}$ such that

$$A_{22} = \overline{A}_{22} \cdot R,$$

$$D(\lambda) \cdot \begin{bmatrix} A_{11}-\lambda I_n & A_{12} & 0 \\ A_{21} & -I_m & \overline{A}_{22} \\ 0 & A_{22} & 0 \\ B_1 & B_2 & 0 \end{bmatrix} = I_{n+m+k}, \tag{14}$$

where $\overline{A}_{22} \in \mathbb{R}^{m\times k}$ and $\operatorname{rank} \overline{A}_{22} = k$.

Step 3. $x(t) = 0,\ y(t) = 0$ for $t \geq 0$.

Proof. By (13) and step 2 we have

$$D(\lambda) \cdot \begin{bmatrix} -x(0) \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \\ R\cdot\hat{\hat{y}}(\lambda) \end{pmatrix}. \tag{15}$$

The function on the left-hand side is of exponential type zero. Hence it follows from the theorem of Paley and Wiener that $\hat{x}(\lambda) = 0$ and $\hat{y}(\lambda) = 0$, thus $x(t) = 0$ and $y(t) = 0$ for $0 \le t \le h$.

Discontinuous Part

For a discontinuous part of a solution of System (1), we have that discontinuities take place at $t = t_i + jh,\ t_i \in T_\alpha;\ j = 0, 1, \dots,$ and System (1) reduces to

$$\begin{aligned} x(t) =& 0, \\ y(t) =& A_{22} y(t-h),\ t \ge 0, \\ z(t) =& B_2 y(t),\ t > 0. \end{aligned}$$

Then $z(t_i + h + jh) = B_2 (A_{22})^{j+1} y(t_i), i = 1, \dots, \alpha$ and condition (9) implies $(A_{22})^{0+1} y(t_i) = 0, i = 1, \dots, \alpha$, which concludes the proof of Theorem 2. ■

5. Observability of small solutions with respect to y and x

Now we examine the observability of small solutions with respect to y.

Definition 5 *The nontrivial small solutions with respect to y of System (1) are said to be observable if every nontrivial small solution with respect to y has a nonzero output for some $t \ge 0$. This means that*

$$\exists T > 0 \ \left.\begin{array}{cccccc} y(t) & = 0 & \forall & t & \ge & T \\ z(t) & = 0 & \forall & t & \ge & 0 \end{array}\right\} \Rightarrow y(t) = 0,\ \forall t \ge 0.$$

Theorem 3 *The nontrivial small solutions of System (1) with respect to y are observable if and only if*

$$i) \operatorname{rank} \begin{bmatrix} A_{11} - \lambda I_n \\ A_{21} \\ B_1 \end{bmatrix} < n,\ \textit{for some } \lambda \in \mathbb{C}, \tag{16}$$

$$ii) \operatorname{rank} \begin{bmatrix} -I_m & A_{22} \\ A_{22} & 0 \\ B_2 & 0 \end{bmatrix} = m + \operatorname{rank} A_{22}, \tag{17}$$

$$iii) \ \operatorname{rank} \begin{bmatrix} B_2 \\ B_2 A_{22} \\ \vdots \\ B_2 (A_{22})^{m-1} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B_2 \\ B_2 A_{22} \\ \vdots \\ B_2 (A_{22})^{m-1} \\ A_{22} \end{bmatrix}. \tag{18}$$

Proof.

Necessity

Let us assume that a nontrivial small solution with respect to y exists. Then for any value of a solution x the output is zero. We have three cases here: first when x equals zero, second when x changes into zero after some time, third when x is nonzero. In the third case (the general case) we have

$$\exists \lambda_0 \in \mathbb{C} \ \operatorname{rank} \begin{bmatrix} A_{11} - \lambda_0 I_n \\ A_{21} \end{bmatrix} < n$$

and the condition for the output $z(t) = B_1 x(t) = 0$, for $t > h$ implies:

$$\operatorname{rank} \begin{bmatrix} A_{11} - \lambda_0 I_n \\ A_{21} \\ B_1 \end{bmatrix} < n.$$

Sufficiency

When investigating the solution with respect to y we can split the solution into a continuous part and a discontinuous one. To observe the continuous part of the solution with respect to y we proceed similarly to Continuous Part of Sufficiency proof of Theorem 2. We leave step 1 to the reader and the outline of steps 2 and 3 is given as follows.

Condition (14) becomes:

$$A_{22} = \overline{A}_{22} \cdot R,$$
$$D_y(\lambda) \cdot \begin{bmatrix} A_{11} - \lambda I_n & A_{12} & 0 \\ A_{21} & -I_m & \overline{A}_{22} \\ A_{21} & A_{22} & 0 \\ B_1 & B_2 & 0 \end{bmatrix} = \begin{pmatrix} 0_{n\times n} & 0_{n\times(n+k)} \\ 0_{(n+k)\times n} & I_{m+k} \end{pmatrix},$$

where $\overline{A}_{22} \in \mathbb{R}^{m\times k}$ and $\operatorname{rank} \overline{A}_{22} = k$. Then, we arrive at condition (15):

$$D_y(\lambda) \cdot \begin{bmatrix} -x(0) \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} 0 \\ \hat{y}(\lambda) \\ R \cdot \hat{y}(\lambda) \end{pmatrix}.$$

The function on the left-hand side is of exponential type zero. Hence, it follows from the theorem of Paley and Wiener that $\hat{y}(\lambda) = 0$, thus $y(t) = 0$ for $0 \le t \le h$ and condition (17) holds. Proof of observability of the discontinuous part of the solution with respect to y is the same as the proof of Theorem 2 of sufficiency, discontinuous part. These conclude the proof of Theorem 3. ■

COROLLARY 1 *The nontrivial small solutions of System (1) with respect to y are observable only if nontrivial small solutions of System (1) are observable.*

Proof. If conditions (16), (17) and (18) are satisfied then for $\lambda \notin \sigma(A_{11})$ (8) is true and (9) is the same as (18). However, if condition (8) is true and $\operatorname{rank}\begin{bmatrix} A_{21} \\ B_1 \end{bmatrix} = n$, then (16) is always false. ∎

Now we examine the observability of small solutions with respect to x.

DEFINITION 6 *The nontrivial small solutions with respect to x of System (1) are said to be observable if every small solution with respect to x has a nonzero output for some $t \geq 0$. This means that*

$$\exists T > 0 \quad \left.\begin{matrix} x(t) & = 0 & \forall & t & \geq & T \\ z(t) & = 0 & \forall & t & \geq & 0 \end{matrix}\right\} \Rightarrow x(t) = 0, \ \forall t \geq 0.$$

THEOREM 4 *The nontrivial small solutions of System (1) with respect to x are observable if and only if*

$$\textit{i)} \ \operatorname{rank}\begin{bmatrix} A_{12} & 0 \\ -I_m & A_{22} \\ B_2 & 0 \end{bmatrix} < m + \operatorname{rank} A_{22}, \tag{19}$$

$$\textit{ii)} \operatorname{rank}\begin{bmatrix} A_{11} - \lambda I \\ A_{21} \\ B_1 \end{bmatrix} = n, \ \textit{for some } \lambda \in \mathbb{C}, \tag{20}$$

$$\textit{iii)} \ \operatorname{rank}\begin{bmatrix} B_2 \\ B_2 A_{22} \\ \vdots \\ B_2 (A_{22})^{m-1} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B_2 \\ B_2 A_{22} \\ \vdots \\ B_2 (A_{22})^{m-1} \\ A_{22} \end{bmatrix}. \tag{21}$$

Proof.

Necessity

Let us assume that nontrivial small solutions with respect to x exist. Then for any value of solutions y the output is zero. We have three cases here: first when y equals zero, second when y changes into zero after some time, third when y is nonzero. In the third case (the general case) we have

$$\operatorname{rank}\begin{bmatrix} A_{12} & 0 \\ -I_m & A_{22} \end{bmatrix} < m + \operatorname{rank} A_{22}$$

and the condition for the output $z(t) = B_2 y(t) = 0$, for $t > h$ implies:

$$\operatorname{rank}\begin{bmatrix} A_{12} & 0 \\ -I_m & A_{22} \\ B_2 & 0 \end{bmatrix} < m + \operatorname{rank} A_{22}.$$

Thus, condition (19) holds.

Sufficiency

When investigating the solution with respect to x we can split the solution into a continuous part and a discontinuous one. To observe the continuous part of the solution with respect to x we proceed similarly to Continuous Part of Sufficiency proof of Theorem 2, we leave step 1 to the reader and the outline of steps 2 and 3 is given by the following:

Condition (14) becomes:

$$A_{22} = \overline{A}_{22} \cdot R,$$

$$D_x(\lambda) \cdot \begin{bmatrix} A_{11} - \lambda I_n & A_{12} & 0 \\ A_{21} & -I_m & \overline{A}_{22} \\ 0 & A_{22} & \overline{A}_{22} \\ B_1 & B_2 & 0 \end{bmatrix} = \begin{pmatrix} I_{n\times n} & 0_{n\times(n+k)} \\ 0_{(n+k)\times n} & I_{(n+k)\times(m+k)} \end{pmatrix},$$

where $\overline{A}_{22} \in \mathbb{R}^{m\times k}$ and $\operatorname{rank} \overline{A}_{22} = k$. Then, condition (15) becomes

$$D_x(\lambda) \cdot \begin{bmatrix} -x(0) \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} \hat{x} \\ 0 \\ 0 \end{pmatrix}.$$

The function on the left-hand side is of exponential type zero. Hence it follows from a theorem of Paley and Wiener that $\hat{x}(\lambda) = 0$ thus $x(t) = 0$ for $0 \le t \le h$ and condition (20) holds. Proof of observability of the discontinuous part of the solution with respect to x is the same as the proof of Theorem 2 of sufficiency, discontinuous part. These conclude the proof of Theorem 4. ■

6. Examples

EXAMPLE 3 Consider System (1) composed of the following matrices

$$A_{11} = \begin{pmatrix} 1 \end{pmatrix}, \; A_{12}^T = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; A_{21} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, A_{22} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; B_1 = \begin{pmatrix} -2 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}.$$

Then we check condition (8):

$$\underset{\lambda\in\mathbb{C}}{max} \operatorname{rank} \begin{bmatrix} 1-\lambda & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} = 5$$

and conditions (16), (17) and (9):

$$\operatorname{rank}\begin{bmatrix}1-\lambda\\ 1\\ 1\\ 1\\ -2\end{bmatrix} = 1 = n \text{ for all } \lambda\in\mathbb{C}, \quad \operatorname{rank}\begin{bmatrix}-1 & 0 & 0 & 0 & 0 & 1\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 & 0\end{bmatrix}$$

$$= 3 < m+k,$$

$$0 = \operatorname{rank}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \neq \operatorname{rank}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = 1.$$

Thus, this system observes neither its nontrivial small solutions nor nontrivial small solutions with respect to x and y.

Example 4 Let a system of the form (1) be given by

$$A_{11}=\begin{pmatrix}1\end{pmatrix},\ A_{12}^T=\begin{pmatrix}1\\ 1\end{pmatrix},\ A_{21}=\begin{pmatrix}0\\ 0\end{pmatrix}, A_{22}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\ B_1=\begin{pmatrix}0\end{pmatrix},\ B_2=\begin{pmatrix}3 & 0\end{pmatrix}.$$

It is easy to see that condition (8) is satisfied:

$$\max_{\lambda\in\mathbb{C}} \operatorname{rank}\begin{bmatrix}1-\lambda & 1 & 1 & 0 & 0\\ 0 & -1 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 3 & 0 & 0 & 0\end{bmatrix} = 4,$$

while conditions (16) and (17) take the form

$$\operatorname{rank}\begin{bmatrix}1-\lambda_0\\ 0\\ 0\\ 0\\ 0\end{bmatrix} \overset{\lambda_0=1}{=} 0 < n, \operatorname{rank}\begin{bmatrix}-1 & 0 & 0 & 1\\ 0 & -1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 3 & 0 & 0 & 0\end{bmatrix} = 3 = m+k$$

and conditions (9), (19) are as follows

$$1 = \operatorname{rank} \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \qquad = \operatorname{rank} \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = 1, \ \operatorname{rank} \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$$

$$= 3 \nless m + k = 3.$$

This system satisfies the assumptions of Theorem 3, thus, according to Corollary 1, it also fulfils the assumption of Theorem 2 but does not satisfy the assumptions of Theorem 4.

7. Conclusion

In this paper we investigated the problem of small solutions of linear stationary differential-algebraic systems with a delay. Existence of small solutions, small solutions with respect to both x and y were presented. As a result parametric rank conditions for observability of these types of small solutions were given. Illustrative examples to every theorem were shown. The results obtained can be generalized to differential-algebraic systems with many delays and to more general observability problems for such systems. This will be the object of another paper.

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