Electrical impedance tomography: from topology to shape[*][†]

by

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Abstract: A level set based shape and topology optimization approach to electrical impedance tomography (EIT) problems with piecewise constant conductivities is introduced. The proposed solution algorithm is initialized by using topological sensitivity analysis. Then it relies on the notion of shape derivatives to update the shape of the domains where conductivity takes different values.

Keywords: electrical impedance tomography, inverse problem, shape derivative, topological derivative, level set method.

1. Introduction

Electrical Impedance Tomography (EIT) is a non-destructive imaging technique which has various applications in medical imaging, geophysics and other fields. Its purpose is to reconstruct the electric conductivity and permittivity of hidden objects inside a medium with the help of boundary field measurements. If we denote by Ω the background medium with Σ being its smooth boundary where the currents are applied, then the commonly used continuum model is

$$-\mathrm{div}(q(x,\omega)\nabla u(x,\omega)) = 0 \ \ \text{in } \Omega,$$
$$q(x,\omega)\partial_n u(x,\omega) = f(x,\omega) \ \ \text{on } \Sigma.$$

Here u is the electric potential or voltage, and the admittivity q is given by $q(x,\omega) = \sigma(x,\omega) + i\omega\varepsilon(x,\omega)$, where σ is the electric conductivity, ε is the electric permittivity, and ω is the angular frequency of the applied current. We also need the conservation of charge condition $\int_\Sigma f = 0$ and the condition $\int_\Sigma u = 0$ which

amounts to choosing a "ground" or reference voltage. For further extensions of this continuum model in case of real experiments, we refer to the survey papers of Borcea (2002) and Cheney, Isaacson and Newell (1999).

A widely used solution approach to this inverse problem of identifying q is to minimize the L_2-distance between the potentials u_i pertinent to a certain given number M of applied currents f_i and corresponding measurements m_i. Since the problem is known to be severely ill-posed, it is necessary to add a regularization term in the functional. The resulting minimization problem then becomes

$$\min_q \sum_{i=1}^{M} \int_\Omega (u_i(q) - m_i)^2 + \beta \int_\Omega |\nabla q| \tag{1}$$

where the first term takes care of matching the given measurements and the second term implements the regularization with a positive regularization parameter β. The nondifferentiable nature of this term is well-known to preserve discontinuities, i.e., the interfaces between the background and the inclusions (Chung, Chan and Tai, 2005).

A number of algorithms have been proposed for solving particular situations containing additional information on the data or the underlying configuration. For instance, in Ammari, Moskow and Vogelius (2003), Brühl, Hanke and Vogelius (2003), and Cedio-Fengya, Moskow and Vogelius (1998) it is assumed that the background medium is smooth (with known conductivity) and that it contains a certain number of small inclusions with a higher or lower conductivity. In these papers, the hypothesis of small inhomogeneities allows to perform an asymptotic analysis of the model, which can be used to design different specialized reconstruction algorithms. A similar asymptotic analysis is used in the present paper to produce the so-called topological derivative, which provides information about the location of objects with conductivities different from the background. Another possible hypothesis (or additional a priori information) is to assume that the conductivity q is piecewise constant. In this case, the objects are assumed to have sharp interfaces. Such an assumption allows us to pursue a shape optimization approach, since the regions where the conductivity q is constant define subdomains of the domain Ω; see Chung, Chan and Tai (2005) for a related concept using level set functions for representing the inclusions.

In the present paper we pick up the aforementioned perspective of q being piecewise constant. Therefore we can use the tools of shape and topology optimization as described, e.g., in Sokołowski and Zolésio (1992), Sokołowski and Żochowski (1999). First we introduce the topological derivative (Sokołowski and Zolésio, 1999) for the EIT-problem. We then use this information in an iterative algorithm for initializing the regions with different conductivities. Subsequently, we employ shape sensitiviy for minimizing a least-squares functional on the boundary.

The rest of the paper is organized as follows: In the next section we formulate the problem under consideration, and in Section 3 we introduce the topological

derivative (Sokołowski and Zolésio, 1999) for the EIT problem. The following section is devoted to computing the shape derivative and the derivative with respect to q of the underlying reduced objective functional. Finally, in the last section, we describe the algorithm and provide a few numerical results.

2. The EIT problem

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary Σ. We assume that Ω contains material with electrical conductivity $q(x) \geq q_0 > 0$. Then the electrical potential $u(x)$ satisfies

$$-\mathrm{div}(q\nabla u) = 0 \quad \text{in } \Omega, \tag{2}$$
$$q\partial_n u = f \quad \text{on } \Sigma, \tag{3}$$

where f is an applied current density on Σ satisfying the conservation of charge $\int_\Sigma f(s)\,ds = 0$. The EIT problem consists in finding the electrical conductivity $q(x)$ inside Ω using a set of given values of applied current densities $f_i(x)$, $i = 1, \ldots, M$, on Σ and the corresponding electrical potentials $u_i(x)$ on Σ.

Here we assume that the conductivity is piecewise constant and that it takes two distinct values, q_1 and q_2. Then, Ω can be split into two disjoint domains Ω_1 and Ω_2, with $\Omega = \Omega_1 \cup \Omega_2$ and conductivities q_1 and q_2, respectively, so that $\Sigma \cap \Gamma = \emptyset$ with $\Gamma = \partial\Omega_1$ (see Fig. 1). We then have $q = q_1 \mathbb{1}_{\Omega_1} + q_2 \mathbb{1}_{\Omega_2}$. Due to the particular form of q the regularization term becomes

$$\int_\Omega |\nabla q| = |q_2 - q_1|\mathcal{P}(\Gamma), \tag{4}$$

where $\mathcal{P}(\Gamma)$ stands for the perimeter of Ω_1. Therefore, the problem is reduced to solving the following problem which depends only on Ω_2 and the scalar values q_1, q_2:

$$\text{minimize} \quad \mathcal{J}(\Omega_2, q_1, q_2) = \sum_{i=1}^{M} \int_\Sigma (u_i - m_i)^2 + \beta|q_2 - q_1|\mathcal{P}(\Gamma), \tag{5}$$

where u_i is the solution of

$$-\mathrm{div}(q\nabla u_i) = 0 \quad \text{in } \Omega, \tag{6}$$
$$q\partial_n u_i = f_i \quad \text{on } \Sigma, \tag{7}$$

with f_i a known boundary current density for $i \in \{1, .., M\}$. Further, m_i is the boundary measurement corresponding to f_i. In order to fulfill the compatibility conditions required for the Neumann boundary condition (7), the measurements must satisfy
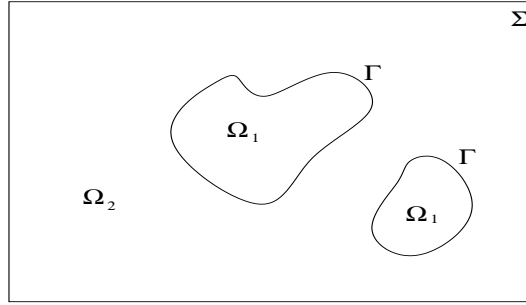
$$\int_\Sigma m_i = 0, \qquad i = 1, .., M. \tag{8}$$

Figure 1. Domain $\Omega = \Omega_1 \cup \Omega_2$.

Since the solution of the Neumann problem (6)-(7) is not uniquely defined, we impose the condition

$$\int_{\Sigma} u_i = 0, \qquad i = 1, .., M, \tag{9}$$

in order to obtain uniqueness. We also introduce the functional

$$J(\Omega_2) = \sum_{i=1}^{M} \int_{\Sigma} (u_i - m_i)^2 + \beta |q_2 - q_1| \mathcal{P}(\Gamma), \tag{10}$$

where q_1, q_2 are now assumed to be fixed. Referring back to (5) we clearly see that it contains Ω_2 as an unknown quantity. Hence, (5) represents a shape optimization problem.

Our subsequent algorithm for solving (5) operates in two steps. First, for fixed q_1, q_2, we minimize $\mathcal{J}$ with respect to Ω_2, i.e., we minimize J. Then, based on the new domain Ω_2, we update q_1, q_2. This cycle is repeated until convergence. In order to accomplish this, in the next two sections we study the minimization of J first without the regularization term in the context of topological sensitivity in Section 3 and then including regularization using tools of shape sensitivity analysis in Section 4. Topological sensitivity is intended to detect the number and positions of the inclusions. Shape sensitivity then "optimizes" the shape of the inclusions.

3. Topological derivative

Now we assume that the domain Ω_1 is a small ball of radius ε and center $\hat{x} = (\hat{x}_1, \hat{x}_2) \in \Omega$, and we write Ω_1^{ε} instead of Ω_1. This allows to perform an asymptotic expansion of the shape functional $J(\Omega_2^{\varepsilon})$ with respect to ε. Here we use $\Omega_2^{\varepsilon} := \Omega \setminus \Omega_1^{\varepsilon}$. Eventually, this provides the topological derivative of J. In what follows, we assume, for the sake of simplicity, that $\hat{x} = (0, 0)$. We also introduce $\Gamma_{\varepsilon} := \partial \Omega_1^{\varepsilon}$. In this simplified framework, we are able to prove that

the solution u_i can be written as $u_i = u^\varepsilon_{1,i}\mathbb{1}_{\Omega^\varepsilon_1} + u^\varepsilon_{2,i}\mathbb{1}_{\Omega^\varepsilon_2}$, with $(u^\varepsilon_{1,i}, u^\varepsilon_{2,i})$ the solution of the coupled system

$$-\Delta u^\varepsilon_{2,i} = 0 \quad \text{in } \Omega^\varepsilon_2, \tag{11}$$

$$q_2\partial_{n_2} u^\varepsilon_{2,i} = f_i \quad \text{on } \Sigma, \tag{12}$$

$$-\Delta u^\varepsilon_{1,i} = 0 \quad \text{in } \Omega^\varepsilon_1, \tag{13}$$

$$u^\varepsilon_{1,i} = u^\varepsilon_{2,i} \quad \text{in } \Gamma_\varepsilon, \tag{14}$$

$$q_2\partial_{n_1} u^\varepsilon_{2,i} = q_1\partial_{n_1} u^\varepsilon_{1,i} \quad \text{on } \Gamma_\varepsilon. \tag{15}$$

Here n_1 and n_2 stand for the outer unit normal vector to Ω^ε_1 and Ω^ε_2, respectively. Thus, on Γ_ε, we have $n_1 = -n_2$. The normal derivatives with respect to n_1 and n_2 are $\partial_{n_1} = \nabla_x \cdot n_1$ and $\partial_{n_2} = \nabla_x \cdot n_2$.

In what follows, for the sake of simplicity, we will drop the subscript i. This corresponds to only one measurement, i.e., $M = 1$. We point out that the case of several measurements is readily deduced from the case $M = 1$.

For the asymptotic expansion, we consider the following problems associated with (11)-(15):

$$-\Delta u^\varepsilon_2 = 0 \quad \text{in } \Omega^\varepsilon_2, \tag{16}$$

$$q_2\partial_{n_2} u^\varepsilon_2 = f \quad \text{on } \Sigma, \tag{17}$$

$$q_2\partial_{n_1} u^\varepsilon_2 = q_1\partial_{n_1} u^\varepsilon_1 \quad \text{on } \Gamma_\varepsilon, \tag{18}$$

and

$$-\Delta u^\varepsilon_1 = 0 \quad \text{in } \Omega^\varepsilon_1, \tag{19}$$

$$u^\varepsilon_1 = u^\varepsilon_2 \quad \text{in } \Gamma_\varepsilon. \tag{20}$$

Problem (16)-(18) is a Neumann problem. Since $\int_\Sigma f = 0$ and

$$\int_{\Gamma_\varepsilon} \partial_{n_1} q_1 u^\varepsilon_1 = \int_{\Omega^\varepsilon_1} \Delta u^\varepsilon_1 = 0,$$

the Neumann problem is compatible. As the solution of (16)-(18) is defined only up to a constant, we impose

$$\int_\Sigma u^\varepsilon_2 = 0 \tag{21}$$

to get uniqueness. We will see that condition (21) has an important contribution in the asymptotic expansion.

3.1. First approximation

The first step of the asymptotic expansion is to approximate u^ε_2 by u_2, the solution of (16)-(18) for $\varepsilon = 0$:

$$-\Delta u_2 = 0 \quad \text{in } \Omega_2, \tag{22}$$

$$q_2\partial_{n_2} u_2 = f \quad \text{on } \Sigma. \tag{23}$$

The compatibility condition for (22)-(23) is satisfied, and the uniqueness of the solution is given by (9). Thus, we introduce the rest $\mathcal{R}_{2,\varepsilon}^1 := u_2^\varepsilon - u_2$ such that

$$u_2^\varepsilon = u_2 + \mathcal{R}_{2,\varepsilon}^1. \tag{24}$$

Then, assuming (local) regularity for the solution u_2 we write the following expansion of u_2 for $x \in \Gamma_\varepsilon$:

$$u_2(x) = u_2(0) + \nabla u_2(0) \cdot x + \frac{1}{2} D^2 u_2(0) x \cdot x + \mathcal{S}_{2,\varepsilon}^1(x) \tag{25}$$

with remainder $\mathcal{S}_{2,\varepsilon}^1(x)$. Now we are looking for an expansion of u_1^ε of the form

$$u_1^\varepsilon = v_0 + v_1 + v_2 + \mathcal{S}_{1,\varepsilon}^1 + \mathcal{R}_{1,\varepsilon}^1. \tag{26}$$

Plugging (24) and (25) into the Dirichlet conditions (20) we get

$$v_0(x) = u_2(0), \tag{27}$$
$$v_1(x) = \nabla u_2(0) \cdot x, \tag{28}$$
$$v_2(x) = \frac{1}{2} D^2 u_2(0) x \cdot x, \tag{29}$$

where $D^2 u_2$ denotes the Hessian of u_2. The quantities $\mathcal{S}_{1,\varepsilon}^1$ and $\mathcal{R}_{1,\varepsilon}^1$ solve

$$-\Delta \mathcal{S}_{1,\varepsilon}^1 = 0 \ \text{ in } \Omega_1^\varepsilon, \tag{30}$$
$$\mathcal{S}_{1,\varepsilon}^1 = \mathcal{S}_{2,\varepsilon}^1 \ \text{ in } \Gamma_\varepsilon, \tag{31}$$

and

$$-\Delta \mathcal{R}_{1,\varepsilon}^1 = 0 \ \text{ in } \Omega_1^\varepsilon, \tag{32}$$
$$\mathcal{R}_{1,\varepsilon}^1 = \mathcal{R}_{2,\varepsilon}^1 \ \text{ in } \Gamma_\varepsilon. \tag{33}$$

3.2.   Second approximation

Now we are looking for an expansion of the term $\mathcal{R}_{2,\varepsilon}^1 = u_2^\varepsilon - u_2$. In what follows, we introduce several Neumann problems to approximate u_2^ε. The solutions of these Neumann problems are unique up to a constant. However, we do not fix the constant immediately, but rather in the last step when we gather all the approximations. First of all we must introduce the Steklov-Poincaré operator or Dirichlet-to-Neumann operator T_{Γ_ε}. Let $z \in H^1(\Omega_2^\varepsilon)$. Then T_{Γ_ε} is defined as $T_{\Gamma_\varepsilon} : H^{\frac{1}{2}}(\Gamma_\varepsilon) \to H^{-\frac{1}{2}}(\Gamma_\varepsilon)$, $z_{|\Gamma_\varepsilon} \mapsto (\partial_{n_1} \hat{z})_{|\Gamma_\varepsilon}$, where $\hat{z}$ is the solution of

$$-\Delta \hat{z} = 0 \ \text{ in } \Omega_1^\varepsilon, \tag{34}$$
$$\hat{z} = z \ \text{ on } \Gamma_\varepsilon. \tag{35}$$

Subtracting (16)-(18) and (22)-(23), we obtain the following equation for $\mathcal{R}_{2,\varepsilon}^1$:

$$-\Delta \mathcal{R}_{2,\varepsilon}^1 = 0 \quad \text{in } \Omega_2^\varepsilon, \tag{36}$$

$$q_2 \partial_{n_2} \mathcal{R}_{2,\varepsilon}^1 = 0 \quad \text{on } \Sigma, \tag{37}$$

$$q_2 \partial_{n_1} \mathcal{R}_{2,\varepsilon}^1 = q_1 \partial_{n_1} u_1^\varepsilon - q_2 \partial_{n_1} u_2 \quad \text{on } \Gamma_\varepsilon. \tag{38}$$

Using (26) in (38) and taking into account that $(\partial_{n_1} \mathcal{R}_{1,\varepsilon}^1)_{|\Gamma_\varepsilon} = T_{\Gamma_\varepsilon}(\mathcal{R}_{2,\varepsilon|\Gamma_\varepsilon}^1)$, which comes from (32)-(33) and the definition of T_{Γ_ε}, we get

$$q_2 \partial_{n_1} \mathcal{R}_{2,\varepsilon}^1 - q_1 T_{\Gamma_\varepsilon}(\mathcal{R}_{2,\varepsilon}^1) = q_1 \partial_{n_1}(v_0 + v_1 + v_2 + \mathcal{S}_{1,\varepsilon}^1) - q_2 \partial_{n_1} u_2 \quad \text{on } \Gamma_\varepsilon. \tag{39}$$

Note that any constant solves the homogeneous problem associated with (36)-(37) and (39). Indeed, if λ is a given constant and $\hat{\lambda}$ is the solution of (34)-(35) with λ on the right-hand side, it is easy to see that $\hat{\lambda} = \lambda$ and, thus, $T_{\Gamma_\varepsilon}(\lambda) = \partial_{n_1}\hat{\lambda} = 0$. Therefore, the solution of (36)-(37) and (39) is unique up to a constant as usual for Neumann problems. Hence, we are looking for a solution $\mathcal{R}_{2,\varepsilon}^1$ of the form

$$\mathcal{R}_{2,\varepsilon}^1 = w_0 + w_1 + w_2 + \mathcal{R}_{2,\varepsilon}^2. \tag{40}$$

Since v_0 is constant, we choose $w_0 \equiv 0$. We also have the following expansion for $\partial_{n_1} u_2$:

$$\partial_{n_1} u_2(x) = \nabla u_2(0) \cdot n_1 + D^2 u_2(0)x \cdot n_1 + \mathcal{S}_{2,\varepsilon}^2(x). \tag{41}$$

Therefore, we define w_1, w_2 and $\mathcal{R}_{2,\varepsilon}^2$ such that the conditions (36)-(37) and

$$q_2 \partial_{n_1} w_1 - q_1 T_{\Gamma_\varepsilon}(w_1) = (q_1 - q_2)\partial_{n_1} u_2(0) \quad \text{on } \Gamma_\varepsilon, \tag{42}$$

$$q_2 \partial_{n_1} w_2 - q_1 T_{\Gamma_\varepsilon}(w_2) = \varepsilon(q_1 - q_2)\left[D^2 u_2(0)n_1 \cdot n_1\right] \quad \text{on } \Gamma_\varepsilon, \tag{43}$$

$$q_2 \partial_{n_1} \mathcal{R}_{2,\varepsilon}^2 - q_1 T_{\Gamma_\varepsilon}(\mathcal{R}_{2,\varepsilon}^2) = q_1 \mathcal{S}_{1,\varepsilon}^1 - q_2 \mathcal{S}_{2,\varepsilon}^2 \quad \text{on } \Gamma_\varepsilon \tag{44}$$

hold true.

3.3. Approximation of w_1

In order to have an approximation of w_1, we use the change of variable $x =: \varepsilon\xi$ and define $\tilde{w}_1(\xi)$ as the solution of

$$-\Delta_\xi \tilde{w}_1 = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{B(0,1)}, \tag{45}$$

$$q_2 \partial_{n_1} \tilde{w}_1 - q_1 T_{\Gamma_\varepsilon}(\tilde{w}_1) = (q_1 - q_2)\partial_{n_1} u_2(0) \quad \text{on } \partial B(0,1). \tag{46}$$

Note that in (46) ∂_{n_1} denotes the operator $\nabla_x \cdot n_1$, which is easily related to the gradient with respect to ξ by $\varepsilon \nabla_x = \nabla_\xi$. For convenience, in what follows we use

$$\alpha := \frac{q_1 - q_2}{\frac{q_1}{N-1} + q_2} \quad \text{and} \quad r = |x| = \varepsilon|\xi|. \tag{47}$$

We have the following proposition.

PROPOSITION 1 *The solution $\tilde{w}_1(\xi)$ to (45)-(46) is given by*

$$\tilde{w}_1(\xi) = -\frac{\varepsilon\alpha}{N-1}\nabla u_2(0) \cdot \frac{\xi}{|\xi|^N}. \tag{48}$$

We further have $\lim_{|\xi|\to\infty} \tilde{w}_1(\xi) = 0$.

Proof. First of all $\tilde{w}_1(\xi)$ given by (48) is harmonic, since it can be checked easily that

$$\Delta_\xi \frac{\xi_i}{|\xi|^N} = 0, \ i = 1,..,N.$$

Therefore, (45) is fulfilled. Now we check the boundary condition (46):

$$q_2 \cdot \partial_{n_1}\tilde{w}_1(\xi)|_{\xi\in\partial B(0,1)} = q_2 \cdot \partial_r \left(-\frac{\varepsilon^N \alpha}{N-1}\frac{1}{r^{N-1}}\nabla u_2(0) \cdot n_1\right)|_{r=\varepsilon}$$

$$= q_2 \cdot \left(\frac{\varepsilon^N \alpha}{r^N}\nabla u_2(0) \cdot n_1\right)|_{r=\varepsilon}$$

$$= \alpha q_2 \partial_{n_1} u_2(0).$$

The solution $\hat{w}_1$ of (34)-(35) with $\tilde{w}_1$ on the right-hand side is given by

$$\hat{w}_1 = -\frac{\alpha\varepsilon}{N-1}\nabla u_2(0)n_1\frac{r}{\varepsilon},$$

and, thus,

$$q_1 \cdot T_{\Gamma_\varepsilon}(\tilde{w}_1) = q_1 \cdot \partial_{n_1}\hat{w}_1|_{\Gamma_\varepsilon} = -\frac{\alpha q_1}{N-1}\partial_{n_1}u_2(0).$$

Finally we get

$$(q_2 \cdot \partial_{n_1}\tilde{w}_1(\xi) - q_1 \cdot T_{\Gamma_\varepsilon}(\tilde{w}_1(\xi)))|_{\xi\in\partial B(0,1)} = (q_1 - q_2)\partial_{n_1}u_2(0).$$

Thus we have proved that the boundary condition (46) is fulfilled.   ■

The approximation $\tilde{w}_1$ does not satisfy the boundary condition (37). Thus, it leaves a discrepancy on Σ which will be compensated with the help of the function z_1 defined in the following way:

$$-\Delta z_1 = 0 \ \text{ in } \Omega_2, \tag{49}$$

$$\partial_{n_2}z_1 = \frac{\varepsilon^N \alpha}{N-1}\partial_{n_2}\left(\nabla u_2(0) \cdot \frac{x}{|x|^N}\right) \ \text{ on } \Sigma. \tag{50}$$

It can be checked for the Neumann problem (49)-(50) that the compatibility condition is fulfilled. Therefore, we have obtained the expansion

$$w_1 = \tilde{w}_1 + z_1 + W_1, \tag{51}$$

where W_1 is defined as $W_1 = w_1 - \tilde{w}_1 - z_1$.

3.4. Approximation of w_2

We are now looking for a similar expansion of w_2. Using the change of variable $x = \varepsilon\xi$ we introduce $\tilde{w}_2(\xi)$ as the solution of

$$-\Delta_\xi \tilde{w}_2 = 0 \quad \text{in} \ \ \mathbb{R}^N \setminus \overline{B(0,1)}, \tag{52}$$

$$q_2 \partial_{n_1} \tilde{w}_2 - q_1 T_{\Gamma_\varepsilon}(\tilde{w}_2) = \varepsilon(q_1 - q_2) \left[D^2 u_2(0) n_1 \cdot n_1 \right] \quad \text{on} \ \ \partial B(0,1). \tag{53}$$

For convenience, in what follows we use

$$\beta := \frac{q_1 - q_2}{\frac{2q_1}{N} + q_2}. \tag{54}$$

Similarly as for $\tilde{w}_1(\xi)$, we find that a solution $\tilde{w}_2(\xi)$ to (52)- (53) is given by

$$\tilde{w}_2(\xi) = -\frac{\varepsilon^2 \beta}{N} \left[\frac{D^2 u_2(0)\xi \cdot \xi}{|\xi|^{N+2}} \right] \tag{55}$$

which tends to 0 as $|\xi| \to \infty$. We introduce the function z_2 which compensates for the discrepancy left by $\tilde{w}_2$ on Σ:

$$-\Delta z_2 = 0 \ \ \text{in} \ \Omega_2, \tag{56}$$

$$\partial_{n_2} z_2 = \frac{\varepsilon^{N+2}\beta}{N} \partial_{n_2} \left[\frac{D^2 u_2(0)x \cdot x}{|x|^{N+2}} \right] \ \ \text{on} \ \Sigma. \tag{57}$$

It can be checked for the Neumann problem (56)-(57) that the compatibility condition is fulfilled. Therefore we have obtained the expansion

$$w_2 = \tilde{w}_2 + z_2 + W_2, \tag{58}$$

where W_2 is defined as $W_2 = w_2 - \tilde{w}_2 - z_2$.

3.5. Uniqueness of u_2^ε

As mentioned earlier, the solutions of (45)-(46), (49)-(50) and (52)-(53), (56)-(57) are not unique. Hence, we must choose a normalisation in order to get uniqueness. For z_1 and z_2 we take $\int_\Sigma z_1 = 0$ and $\int_\Sigma z_2 = 0$. Then we introduce a constant λ_ε and gather (24), (40), (51) and (58) so that we obtain the following expansion for u_2^ε:

$$u_2^\varepsilon(x) = u_2(x) + \tilde{w}_1(\xi) + z_1(x) + \tilde{w}_2(\xi) + z_2(x) + \lambda_\varepsilon$$
$$+ W_1(x) + W_2(x) + \mathcal{R}_{2,\varepsilon}^2. \tag{59}$$

The constant λ_ε must be determined by the normalization condition (21) to guarantee uniqueness of the solution. In view of (59), this condition implies

$$\int_\Sigma u_2(x) + \tilde{w}_1(\xi) + z_1(x) + \tilde{w}_2(\xi) + z_2(x) + \lambda_\varepsilon + W_1(x) + W_2(x) + \mathcal{R}_{2,\varepsilon}^2 = 0.$$

Due to $\int_\Sigma u_2 = 0$ we have

$$\lambda_\varepsilon = -|\Sigma|^{-1} \int_\Sigma \tilde{w}_1(\xi) + \tilde{w}_2(\xi) + W_1(x) + W_2(x) + \mathcal{R}_{2,\varepsilon}^2.$$

For numerical purposes, we split λ_ε into $\lambda_\varepsilon = \lambda_{1,\varepsilon} + \lambda_{2,\varepsilon}$ with

$$\lambda_{1,\varepsilon} = -|\Sigma|^{-1} \int_\Sigma \tilde{w}_1(\xi) + \tilde{w}_2(\xi),$$

$$\lambda_{2,\varepsilon} = -|\Sigma|^{-1} \int_\Sigma W_1(x) + W_2(x) + \mathcal{R}_{2,\varepsilon}^2.$$

Then the main approximation in our numerical approach is

$$u_2^\varepsilon(x) \simeq u_2(x) + \tilde{w}_1(\xi) + z_1(x) + \tilde{w}_2(\xi) + z_2(x) + \lambda_{1,\varepsilon}. \tag{60}$$

3.6.   Expansion of the shape functional $J(\Omega_2)$

As noted before, we consider here the case of a single measurement, i.e. $M = 1$, and we drop the subscript i in all notation. In addition, we assume that $\beta = 0$ in order to neglect the perimeter term which is problematic for topological sensitivity. Rather we capture this by shape sensitivity later. It is necessary to invoke the latter assumption for the expansion of the functional to be of order ε^2. Hence, the shape functional J evaluated at Ω_2^ε is equal to

$$J(\Omega_2^\varepsilon) = \int_\Sigma (u_2^\varepsilon - m)^2. \tag{61}$$

Using (59) in (61) we get

$$J(\Omega_2^\varepsilon) = J(\Omega) + K_{\tilde{w}_1} + K_{\tilde{w}_2} + K_{z_1} + K_{z_2} + K_{\lambda_{1,\varepsilon}} + L_0 + L_1 + L_2,$$

where K_ϕ denotes the integral $K_\phi = \int_\Sigma 2\phi(x)(u_2(x) - m(x))\, dx$ and

$$L_0 = \int_\Sigma (\tilde{w}_1(\xi) + z_1(x) + \tilde{w}_2(\xi) + z_2(x) + \lambda_{1,\varepsilon})^2\, dx,$$

$$L_1 = \int_\Sigma 2(\lambda_{2,\varepsilon} + W_1(x) + W_2(x) + \mathcal{R}_{2,\varepsilon}^2(x))$$
$$\cdot (u_2 - m + \tilde{w}_1(\xi) + z_1(x) + \tilde{w}_2(\xi) + z_2(x) + \lambda_{1,\varepsilon})\, dx,$$

$$L_2 = \int_\Sigma (\lambda_{2,\varepsilon} + W_1(x) + W_2(x) + \mathcal{R}_{2,\varepsilon}^2(x))^2\, dx.$$

Now we analyse each term separately. We start by noting that in view of (8)-(9) we have $K_{\lambda_{1,\varepsilon}} = 0$. Next we introduce the adjoint state p as solution of

$$-\Delta p = 0 \quad \text{in } \Omega, \tag{62}$$
$$\partial_{n_2} p = 2(u_2 - m) \quad \text{on } \Sigma. \tag{63}$$

Note that the compatibility condition for p is obviously satisfied.

3.6.1. Calculation of $K_{\tilde{w}_1} + K_{z_1}$ and $K_{\tilde{w}_2} + K_{z_2}$

Using Green's formula, we find

$$\int_{\Omega_2^\varepsilon} -\Delta p(x)\,\tilde{w}_1(\xi) + p(x)\,\Delta\tilde{w}_1(\xi)\,dx = \int_{\Sigma\cup\Gamma_\varepsilon} -\partial_{n_2}p(x)\,\tilde{w}_1(\xi) + p(x)\,\partial_{n_2}\tilde{w}_1(\xi)\,dx$$
$$= 0$$

and, thus, we get

$$K_{\tilde{w}_1} = \int_\Sigma \partial_{n_2}p(x)\,\tilde{w}_1(\xi)\,dx$$
$$= -\int_{\Gamma_\varepsilon} \partial_{n_2}p(x)\,\tilde{w}_1(\xi)\,dx + \int_{\Gamma_\varepsilon} p(x)\,\partial_{n_2}\tilde{w}_1(\xi)\,dx + \int_\Sigma p(x)\,\partial_{n_2}\tilde{w}_1(\xi)\,dx.$$

Now we compute $\partial_{n_2}\tilde{w}_1(\xi)|_{\Gamma_\varepsilon}$. Note that according to (48) and $x = \varepsilon\xi$ we have

$$\tilde{w}_1(\xi) = -\frac{\varepsilon^N\alpha}{N-1}\nabla u_2(0)\cdot\frac{x}{|x|^N}.$$

Using $\nabla_x(x_i|x|^{-N}) = -Nx_i|x|^{-N-2}x + |x|^{-N}\nabla_x x_i$, we get

$$\partial_{n_2}\tilde{w}_1(\xi) = -\frac{\varepsilon^N\alpha}{N-1}\nabla_x\left(\sum_{i=1}^N \partial_i u_2(0)x_i|x|^{-N}\right)\cdot n_2$$
$$= -\frac{\varepsilon^N\alpha}{N-1}\left(-N|x|^{-N-2}(x\cdot\nabla_x u_2(0))x + |x|^{-N}\nabla_x u_2(0)\right)\cdot n_2.$$

Since $x = -\varepsilon n_2$ on Γ_ε, we obtain

$$\partial_{n_2}\tilde{w}_1(\xi)|_{\Gamma_\varepsilon} = -\frac{\varepsilon^N\alpha}{N-1}\left(-\varepsilon^{-N}N\nabla_x u_2(0)\cdot n_2 + \varepsilon^{-N}\nabla_x u_2(0)\cdot n_2\right)$$
$$= \alpha\nabla_x u_2(0)\cdot n_2$$
$$= -\varepsilon^{-1}\alpha\nabla_x u_2(0)\cdot x.$$

Further we expand p about $x = 0$:

$$\int_{\Gamma_\varepsilon} p(x)\,\partial_{n_2}\tilde{w}_1(\xi)\,dx = \int_{\Gamma_\varepsilon}\left[p(0) + \nabla p(0)\cdot x + O(\varepsilon^2)\right]\cdot\left[-\alpha\varepsilon^{-1}\nabla u_2(0)\cdot x\right]dx.$$

Due to the symmetry of the sphere Γ_ε, we have $\int_{\Gamma_\varepsilon} p(0)\nabla u_2(0)\cdot x = 0$ and

$$\int_{\Gamma_\varepsilon} x_j^2\,dx = N^{-1}\sum_{i=1}^N\int_{\Gamma_\varepsilon} x_i^2\,dx \qquad \forall j\in\{1,..,N\}.$$

Finally, combining our previous results we get

$$\int_{\Gamma_\varepsilon} -\alpha\varepsilon^{-1}\nabla p(0) \cdot x\nabla u_2(0) \cdot x\, dx = -\alpha\varepsilon^{-1}\sum_{i=1}^{N}\partial_i p(0)\partial_i u_2(0)N^{-1}\varepsilon^2|S_\varepsilon^{N-1}|$$
$$= -\varepsilon|S_\varepsilon^{N-1}|\alpha N^{-1}\nabla p(0) \cdot \nabla u_2(0).$$

The computation of the integral thus gives

$$\int_{\Gamma_\varepsilon} p(x)\,\partial_{n_2}\tilde{w}_1(\xi)\, dx = -\varepsilon|S_\varepsilon^{N-1}|\alpha N^{-1}\nabla p(0) \cdot \nabla u_2(0) + o(\varepsilon|S_\varepsilon^{N-1}|).$$

Note that we have $|S_\varepsilon^{N-1}| = 2\pi\varepsilon$ for $N = 2$ and $|S_\varepsilon^{N-1}| = 4\pi\varepsilon^2$ for $N = 3$. By a similar calculation we obtain

$$\int_{\Gamma_\varepsilon} \partial_{n_2} p(x)\,\tilde{w}_1(\xi)\, dx = \varepsilon|S_\varepsilon^{N-1}|\alpha N^{-1}\nabla p(0) \cdot \nabla u_2(0) + o(\varepsilon|S_\varepsilon^{N-1}|).$$

For z_1, we conclude

$$K_{z_1} = \int_{\Sigma} \partial_{n_2} p(x)\, z_1(x)\, dx$$
$$= -\int_{\Gamma_\varepsilon} \partial_{n_2} p(x)\, z_1(x)\, dx + \int_{\Gamma_\varepsilon} p(x)\,\partial_{n_2} z_1(x)\, dx + \int_{\Sigma} p(x)\,\partial_{n_2} z_1(x)\, dx.$$

The definition (49)-(50) of z_1 yields $\partial_{n_2} z_1(x) = -\partial_{n_2}\tilde{w}_1(\xi)$ on Σ, and, thus,

$$\int_{\Sigma} p(x)\,\partial_{n_2} z_1(x)\, dx + \int_{\Sigma} p(x)\,\partial_{n_2}\tilde{w}_1(\xi)\, dx = 0. \tag{64}$$

Then we obtain

$$K_{\tilde{w}_1} + K_{z_1} = -2\varepsilon|S_\varepsilon^{N-1}|\alpha N^{-1}\nabla p(0) \cdot \nabla u_2(0) + o(\varepsilon|S_\varepsilon^{N-1}|)$$
$$- \int_{\Gamma_\varepsilon} \partial_{n_2} p(x)\, z_1(x)\, dx + \int_{\Gamma_\varepsilon} p(x)\,\partial_{n_2} z_1(x)\, dx. \tag{65}$$

A similar calculation yields

$$K_{\tilde{w}_2} + K_{z_2} = -\int_{\Gamma_\varepsilon} \partial_{n_2} p(x)\, z_2(x)\, dx + \int_{\Gamma_\varepsilon} p(x)\,\partial_{n_2} z_2(x)\, dx$$
$$+ o(\varepsilon|S_\varepsilon^{N-1}|). \tag{66}$$

3.6.2. Calculation of L_0

We split L_0 into $L_0 = L_0^{(1)} + L_0^{(2)} + L_0^{(3)}$ with

$$L_0^{(1)} = \int_\Sigma (\tilde{w}_1(\xi) + \tilde{w}_2(\xi) + \lambda_{1,\varepsilon})^2,$$

$$L_0^{(2)} = \int_\Sigma (z_1(x) + z_2(x))^2,$$

$$L_0^{(3)} = \int_\Sigma 2(\tilde{w}_1(\xi) + \tilde{w}_2(\xi) + \lambda_{1,\varepsilon})(z_1(x) + z_2(x)).$$

For $L_0^{(1)}$ we get

$$L_0^{(1)} = \int_\Sigma \tilde{w}_1(\xi)^2 + \tilde{w}_2(\xi)^2 + \lambda_{1,\varepsilon}^2 + 2(\tilde{w}_1(\xi)\tilde{w}_2(\xi)) + 2(\tilde{w}_1(\xi) + \tilde{w}_2(\xi))\lambda_{1,\varepsilon}$$

$$= -|\Sigma|\lambda_{1,\varepsilon}^2 + \int_\Sigma \tilde{w}_1(\xi)^2 + \int_\Sigma \tilde{w}_2(\xi)^2 + 2\int_\Sigma \tilde{w}_1(\xi)\tilde{w}_2(\xi).$$

Further, we obtain with $x = (x_1, x_2, ..., x_N)$ and according to (48)

$$\int_\Sigma \tilde{w}_1(\xi)^2 = \varepsilon^{2N} \frac{\alpha^2}{(N-1)^2} \sum_{i,j=1}^{N} \partial_i u_2(0)\partial_j u_2(0) I_{i,j}^{(1)},$$

$$\int_\Sigma \tilde{w}_2(\xi)^2 = \frac{\varepsilon^{2(N+2)}\beta^2}{(N)^2} \sum_{i,j,k,l=1}^{N} \partial_{ij}^2 u_2(0)\partial_{k,l}^2 u_2(0) I_{i,j,k,l}^{(2)},$$

$$\int_\Sigma \tilde{w}_1(\xi)\tilde{w}_2(\xi) = \frac{\varepsilon^{2N+2}\alpha\beta}{(N-1)N} \sum_{i,j,k=1}^{N} \partial_k u_2(0)\partial_{ij}^2 u_2(0) I_{i,j,k}^{(12)}$$

with

$$I_{i,j}^{(1)} = \int_\Sigma \frac{x_i x_j}{|x|^{2N}}, \qquad I_{i,j,k,l}^{(2)} = \int_\Sigma \frac{x_i x_j x_k x_l}{|x|^{2(N+2)}}, \qquad I_{i,j,k}^{(12)} = \int_\Sigma \frac{x_i x_j x_k}{|x|^{2N+2}}. \quad (67)$$

For $\lambda_{1,\varepsilon}$ we compute

$$\lambda_{1,\varepsilon}^2 = |\Sigma|^{-2} \left[\int_\Sigma \tilde{w}_1(\xi) + \tilde{w}_2(\xi) \right]^2$$

$$= |\Sigma|^{-2} \left[\frac{\varepsilon^N \alpha}{N-1} \sum_{k=1}^{N} \partial_k u_2(0) I_k^{(\lambda,1)} + \frac{\varepsilon^{N+2}\beta}{N} \sum_{i,j=1}^{N} \partial_{i,j}^2 u_2(0) I_{i,j}^{(\lambda,2)} \right]^2$$

with

$$I_k^{(\lambda,1)} = \int_\Sigma \frac{x_k}{|x|^N}, \qquad I_{i,j}^{(\lambda,2)} = \int_\Sigma \frac{x_i x_j}{|x|^{N+2}}. \quad (68)$$

3.6.3.   Expansion of the functional

In order to underline the dependence on $\hat{x}$ of the different quantities, in this section, we do no longer assume that $\hat{x} = (0,0)$ as it was the case in our previous calculations. Gathering the previous results, we obtain the following expansion for the functional $J(\Omega_2^\varepsilon)$:

$$J(\Omega_2^\varepsilon) = J(\Omega) + \sum_{k=0}^{4} \mathcal{T}_{k,\varepsilon}(\hat{x}) + o(\varepsilon|S_\varepsilon^{N-1}|) + \mathcal{Z} + \mathcal{Y},$$

with

$$\mathcal{T}_{0,\varepsilon}(\hat{x}) = -2\varepsilon|S_\varepsilon^{N-1}|\alpha N^{-1}\nabla p(\hat{x})\nabla u_2(\hat{x}),$$

$$\mathcal{T}_{1,\varepsilon}(\hat{x}) = \frac{\varepsilon^{2N}\alpha^2}{(N-1)^2} \sum_{i,j=1}^{N} \partial_i u_2(\hat{x})\partial_j u_2(\hat{x})I_{i,j}^{(1)},$$

$$\mathcal{T}_{2,\varepsilon}(\hat{x}) = \frac{\varepsilon^{2(N+2)}\beta^2}{(N)^2} \sum_{i,j,k,l=1}^{N} \partial_{ij}^2 u_2(\hat{x})\partial_{k,l}^2 u_2(\hat{x})I_{i,j,k,l}^{(2)},$$

$$\mathcal{T}_{3,\varepsilon}(\hat{x}) = \frac{2\varepsilon^{2N+2}\alpha\beta}{(N-1)N} \sum_{i,j,k=1}^{N} \partial_k u_2(\hat{x})\partial_{ij}^2 u_2(\hat{x})I_{i,j,k}^{(12)},$$

$$\mathcal{T}_{4,\varepsilon}(\hat{x}) = -|\Sigma|^{-1}\left[\frac{\varepsilon^N\alpha}{N-1}\sum_{k=1}^{N}\partial_k u_2(\hat{x})I_k^{(\lambda,1)} + \frac{\varepsilon^{N+2}\beta}{N}\sum_{i,j=1}^{N}\partial_{i,j}^2 u_2(\hat{x})I_{i,j}^{(\lambda,2)}\right]^2,$$

where $I_{i,j}^{(1)}$, $I_{i,j,k,l}^{(2)}$, $I_{i,j,k}^{(12)}$ and $I_{i,j}^{(\lambda,2)}$ are defined in (67) and (68). We also define

$$\mathcal{Y} = L_1 + L_2, \tag{69}$$

and

$$\mathcal{Z} = -\int_{\Gamma_\varepsilon} \partial_{n_2}p(x)\, z_1(x)\, dx + \int_{\Gamma_\varepsilon} p(x)\, \partial_{n_2}z_1(x)\, dx$$
$$-\int_{\Gamma_\varepsilon} \partial_{n_2}p(x)\, z_2(x)\, dx + \int_{\Gamma_\varepsilon} p(x)\, \partial_{n_2}z_2(x)\, dx + L_0^{(2)} + L_0^{(3)}.$$

4.   Shape derivative

Consider the problems (16)-(18) and (19)-(20) in the domains Ω_1 and Ω_2, i.e.,

$$-\Delta u_2 = 0 \quad \text{in } \Omega_2, \tag{70}$$
$$q_2\partial_{n_2}u_2 = f \quad \text{on } \Sigma, \tag{71}$$
$$q_2\partial_{n_1}u_2 = q_1\partial_{n_1}u_1 \quad \text{on } \Gamma \tag{72}$$

and

$$-\Delta u_1 = 0 \quad \text{in } \Omega_1, \tag{73}$$
$$u_1 = u_2 \quad \text{in } \Gamma. \tag{74}$$

Let $V : \mathbb{R}^N \to \mathbb{R}^N$ be a given smooth vector field with compact support in $\mathbb{R}^N$ According to Sokołowski and Zolésio (1992), the shape derivatives u_1' and u_2' of u_1 and u_2, respectively, solve the following systems

$$-\Delta u_2' = 0 \quad \text{in } \Omega_2, \tag{75}$$
$$q_2 \partial_{n_2} u_2' = 0 \quad \text{on } \Sigma, \tag{76}$$

$$q_2 \partial_{n_2} u_2' = q_1 \partial_{n_2} u_1' - \frac{\partial^2}{\partial n_2^2}(q_2 u_2 - q_1 u_1) v_{n_2},$$
$$+ \nabla_\Gamma (q_2 u_2 - q_1 u_1) \cdot \nabla_\Gamma v_{n_2} \quad \text{on } \Gamma \tag{77}$$

and further

$$-\Delta u_1' = 0 \quad \text{in } \Omega_1, \tag{78}$$
$$u_1' = \partial_{n_1}(u_2 - u_1) v_{n_1} + u_2' \quad \text{in } \Gamma, \tag{79}$$

where the tangential gradient ∇_Γ is defined as $\nabla_\Gamma = \nabla - \partial_{n_i} n_i$, and $v_{n_i} = V \cdot n_i$, $i = 1, 2$.

4.1. First-order shape derivative of the functional

The derivative of functional (10) with respect to the shape is

$$dJ(\Omega_2, V) = \int_\Sigma 2(u_2 - m) u_2' + \beta |q_2 - q_1| \int_\Gamma \mathcal{H}_2 v_{n_2}, \tag{80}$$

where $\mathcal{H}_2$ is the curvature of the boundary of Ω_2. Next we introduce the following adjoint states p_1 and p_2 as the solutions to

$$-\Delta p_2 = 0 \quad \text{in } \Omega_2, \tag{81}$$
$$\partial_{n_2} p_2 = -\partial_{n_1} p_1 \quad \text{on } \Gamma, \tag{82}$$
$$\partial_{n_2} p_2 = 2(u_2 - m) \quad \text{on } \Sigma \tag{83}$$

and

$$-\Delta p_1 = 0 \quad \text{in } \Omega_2, \tag{84}$$
$$p_1 = \frac{q_1}{q_2} p_2 \quad \text{on } \Gamma, \tag{85}$$

respectively. Hence, we obtain

$$0 = \int_{\Omega_2} -u_2'\Delta p_2 + p_2 \Delta u_2' = \int_\Gamma -u_2'\partial_{n_2} p_2 + \partial_{n_2} u_2' p_2, \tag{86}$$

$$0 = \int_{\Omega_1} -u_1'\Delta p_1 + p_1 \Delta u_1' = \int_\Gamma -u_1'\partial_{n_1} p_1 + p_1 \partial_{n_1} u_1'. \tag{87}$$

In view of the boundary conditions in (75)-(79) and (81)-(85) we get

$$\int_\Sigma 2(u_2 - m)u_2' = \int_\Gamma -u_2'\partial_{n_2} p_2 + \partial_{n_2} u_2' p_2 - u_1'\partial_{n_1} p_1 + p_1 \partial_{n_1} u_1'.$$

Now using (77) and (79) we obtain

$$\int_\Sigma 2(u_2 - m)u_2' = \int_\Gamma -u_2'\partial_{n_2} p_2 - u_2'\partial_{n_1} p_1 - \partial_{n_1} p_1 \partial_{n_1}(u_2 - u_1)v_{n_1}$$

$$+ \int_\Gamma p_1 \partial_{n_1} u_1' + \frac{q_1}{q_2} p_2 \partial_{n_1} u_1' - \frac{p_2}{q_2}\frac{\partial^2}{\partial n_2^2}(q_2 u_2 - q_1 u_1)v_{n_2}$$

$$+ \int_\Gamma \frac{p_2}{q_2}\nabla_\Gamma(q_2 u_2 - q_1 u_1) \cdot \nabla_\Gamma v_{n_2}.$$

In view of (83) and (85) we finally conclude

$$\int_\Sigma 2(u_2 - m)u_2' = \int_\Gamma -\partial_{n_1} p_1 \partial_{n_1}(u_2 - u_1)v_{n_1} + \int_\Gamma -\frac{p_2}{q_2}\frac{\partial^2}{\partial n_2^2}(q_2 u_2 - q_1 u_1)v_{n_2}$$

$$+ \int_\Gamma \frac{p_2}{q_2}\nabla_\Gamma(q_2 u_2 - q_1 u_1) \cdot \nabla_\Gamma v_{n_2}.$$

In order to further process the right hand side above, we rely on the following proposition; see Henrot and Pierre (2005, Proposition 5.4.12).

PROPOSITION 2 *Let Ω be an open set of class C^2 and $u : \overline{\Omega} \to \mathbb{R}$ of class C^2. Let n be the outer unit normal vector to Ω and $\mathcal{H}$ the curvature of $\Gamma = \partial\Omega$. Then*

$$\Delta u = \Delta_\Gamma u + \mathcal{H}\partial_{n_2} u + \frac{\partial^2 u}{\partial n_2^2} \text{ on } \Gamma,$$

where Δ_Γ is the so-called Laplace-Beltrami operator on Γ.

Since $\Delta u_1 = \Delta u_2 = 0$ and $\partial_{n_1}(q_2 u_2 - q_1 u_1) = 0$ on Γ, we have

$$\int_\Sigma 2(u_2 - m)u_2' = \int_\Gamma -\partial_{n_1} p_1 \partial_{n_1}(u_2 - u_1)v_{n_1}$$

$$+ \int_\Gamma \frac{p_2}{q_2}[\nabla_\Gamma(q_2 u_2 - q_1 u_1) \cdot \nabla_\Gamma v_{n_2} + \Delta_\Gamma(q_2 u_2 - q_1 u_1)v_{n_2}]$$

$$= \int_\Gamma -\partial_{n_1} p_1 \partial_{n_1}(u_2 - u_1)v_{n_1}$$

$$+ \int_\Gamma \frac{p_2}{q_2}\nabla_\Gamma(q_2 u_2 - q_1 u_1) \cdot \nabla_\Gamma v_{n_2} - \nabla_\Gamma(q_2 u_2 - q_1 u_1) \cdot \nabla_\Gamma\left(\frac{p_2}{q_2}v_{n_2}\right),$$

and finally

$$\int_\Sigma 2(u_2 - m)u_2' = \int_\Gamma \left[-\partial_{n_1} p_1 \partial_{n_1}(u_2 - u_1) + \nabla_\Gamma (u_2 - \frac{q_1}{q_2} u_1) \cdot \nabla_\Gamma p_2 \right] v_{n_1}.$$

According to the boundary conditions (72)-(74) and (82)-(85) we get

$$dJ(\Omega_2, V) = \int_\Gamma \left(\frac{q_2}{q_1} - 1 \right) (\partial_{n_1} p_1 \partial_{n_1} u_2 + \nabla_\Gamma u_2 \cdot \nabla_\Gamma p_1) \, v_{n_1} + \beta |q_2 - q_1| \int_\Gamma \mathcal{H}_2 v_{n_2}$$

$$= \int_\Gamma \left[\left(1 - \frac{q_2}{q_1} \right) (\nabla p_1 \cdot \nabla u_2) + \beta |q_2 - q_1| \mathcal{H}_2 \right] v_{n_2}. \qquad (88)$$

4.2.   Derivative with respect to the conductivity q_1

We consider small perturbations of the conductivity q_1 of the form $q_1^\varepsilon = q_1 + \varepsilon q_1'$. Denote by u_1^ε and u_2^ε the solutions of (70)-(74). Formally, we define the derivatives u_1' and u_2' by

$$u_1' = \lim_{\varepsilon \to 0} \frac{u_1^\varepsilon - u_1}{\varepsilon} \quad \text{and} \quad u_2' = \lim_{\varepsilon \to 0} \frac{u_2^\varepsilon - u_2}{\varepsilon}. \qquad (89)$$

Substituting (89) in (70)-(74) we get the following equations for u_1' and u_2':

$$-\Delta u_2' = 0 \ \text{ in } \Omega_2, \qquad (90)$$

$$q_2 \partial_{n_2} u_2' = 0 \ \text{ on } \Sigma, \qquad (91)$$

$$q_2 \partial_{n_1} u_2' = q_1' \partial_{n_1} u_1 + q_1 \partial_{n_1} u_1' \ \text{ on } \Gamma \qquad (92)$$

and

$$-\Delta u_1' = 0 \ \text{ in } \Omega_1, \qquad (93)$$

$$u_1' = u_2' \ \text{ in } \Gamma. \qquad (94)$$

We assume that $q_1 \neq q_2$ as $q_1 = q_2$ leads to a trivial situation. Therefore, the derivative of $\mathcal{J}$ with respect to q_1 is

$$d\mathcal{J}(\Omega_2, q_1, q_2)(q_1') = \int_\Sigma 2(u_2 - m)u_2' + \text{sign}(q_1 - q_2)\beta q_1' \mathcal{P}(\Gamma). \qquad (95)$$

The term $\int_\Sigma 2(u_2 - m)u_2'$ in (95) can be computed in a similar way as in Section 4 introducing the adjoint states (81)-(85). From Green's formula for p_1 and p_2, respectively, we get

$$\int_\Sigma 2(u_2 - m)u_2' = \int_\Sigma \partial_{n_2} p_2 u_2' = \int_\Gamma -u_2' \partial_{n_2} p_2 - u_1' \partial_{n_1} p_1 + p_2 \partial_{n_2} u_2' + p_1 \partial_{n_1} u_1'. \qquad (96)$$

Finally, using the relations (81)-(85) and (90)-(94) we obtain

$$d\mathcal{J}(\Omega_2, q_1, q_2)(q_1') = -\frac{q_1'}{q_2} \int_\Gamma p_2 \partial_{n_1} u_1 + \text{sign}(q_1 - q_2)\beta q_1' \mathcal{P}(\Gamma). \qquad (97)$$

5.   Numerical results

Now we briefly sketch our algorithmic approach and report on numerical results obtained by our combined shape and topology optimization technique.

5.1.   Algorithm

For some $\varepsilon > 0$, the topological derivative $\mathcal{T}_\varepsilon$ is defined as

$$\mathcal{T}_\varepsilon = (\varepsilon |S_\varepsilon^{N-1}|)^{-1} \sum_{k=1}^{4} \mathcal{T}_{k,\varepsilon}.$$

We initialize our algorithmic procedure by $\Omega_2 := \Omega$ and

$$\Omega_1 := \{x \in \Omega \mid \mathcal{T}_\varepsilon(x) < \gamma \min_{y \in \Omega} \mathcal{T}_\varepsilon(y)\},$$

where $0 < \gamma < 1$ is a given threshold.

Exploring a range of pre-selected ε- and γ-values, we choose the parameter combination which gives the lowest value of $\mathcal{J}(\Omega_2, q_1, q_2)$. Using a moderate number of parameter combinations, this start-up procedure is usually quite fast.

After this initialization of the domains by topological sensitivity, the remaining iterations are as follows: First we update the shape using the shape derivative in a steepest descent framework. According to (88) the steepest descent direction is given by

$$v_{n_2} = -\left[\left(1 - \frac{q_2}{q_1}\right)(\nabla p_1 \cdot \nabla u_2) + \beta |q_2 - q_1| \mathcal{H}_2\right] \quad \text{on } \Gamma.$$

For the representation and transport of the geometry, i.e., Ω_1 respectively Ω_2, according to this velocity we use a level set method (Osher and Fedkiw, 2003; Osher and Sethian, 1988). Note that due to the nature of shape sensitivity, the update velocity v_{n_2} normal to the internal boundary of Ω_2 is defined only on Γ. As the transport of domains is achieved by the level set equation

$$\phi_t + V\|\nabla\phi\| = 0 \quad \text{in } \Omega,$$

we have to extend v_{n_2} to the entire domain Ω (this yields V in the above equation). Above, ϕ represents the level set function which is assumed to be a signed distance function with $\phi|_\Gamma = 0$. We extend v_{n_2} constant along normals to Γ. Observe that the level set equation is a partial differential equation of Hamilton-Jacobi-type. For its appropriate numerical treatment we refer to Osher and Fedkiw (2003), Osher and Sethian (1988). Having advanced the geometry, then, in a second step, according to (97), the value of q_1 is updated by

$$q_1' = \frac{1}{q_2} \int_\Gamma p_2 \partial_{n_1} u_1 - \text{sign}(q_1 - q_2)\beta\mathcal{P}(\Gamma)$$

again within a steepest descent framework. For choosing an appropriate step

length, we use a standard Armijo line search procedure with backtracking.

We point out that the shape as well as the q-update procedure might rely on Newton-type updates as well. This, however, requires the additional work of solving elliptic systems on Γ and is subject to future research.

5.2. Results

In our two examples reported below, we use a finite volume method for the discretization of the state equation, with a grid of 50×50 elements. The two examples present different geometries of the conductivity distributions, and different noise levels. In the first example, the geometry of the unknown object is made of three separate balls, and the noise level is 1%; see Fig. 2. In the second example we do not have noise in the data, and the geometry is made of three different objects, one of which is concave; see Fig. 3. In both cases the number of measurements is $M = 60$, and the regularization parameter is $\beta = 3.10^{-8}$. The true values of the conductivities are $q_1 = 1$ and $q_2 = 10$. We initialize the algorithm by $q_1^{init} = 4$ and $q_2^{init} = 10$, i.e., we assume the background conductivity to be known.

In the first example, the value for q_1 upon termination of our algorithm after 2000 iterations is $q_1^{final} = 1.28$. In Fig. 2 we plot the result after 2000 iterations. The left plot depicts the behavior of q_1 over the iterations, while the right plot provides the detection of the inclusion. The geometry after the initialization by topological sensitivity is shown in light gray, the result upon termination of our algorithm is represented in darker gray and the true situation is shown in stripes. In the second example, allowing for 10000 iterations, the algorithm finds $q_1^{final} = 0.96$. Fig. 3 depicts the result after 10000 iterations.
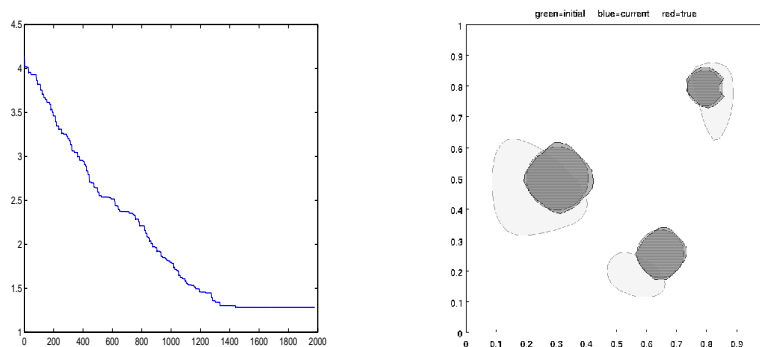

Figure 2. First example after 2000 iterations, with 1% noise. Left: Conductivity q_1 over the iterations. Right: Initial domain Ω_1 after topological sensitivity (light gray); final domain Ω_1 upon termination o the algorithm (darker gray); true domain Ω_1 (stripes).
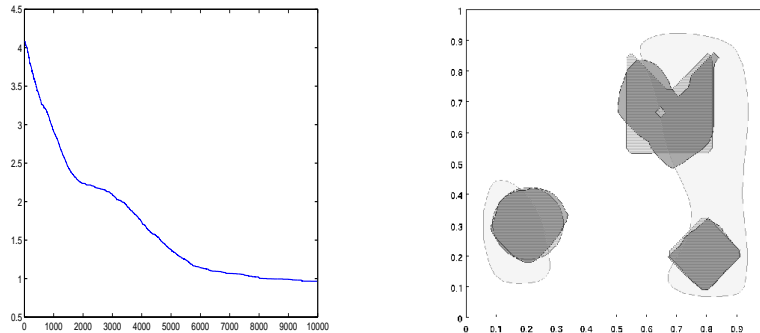
Figure 3. Second example after 10000 iterations. Left: Conductivity q_1 over the iterations. Right: Initial domain Ω_1 after topological sensitivity (light gray); final domain Ω_1 upon termination o the algorithm (darker gray); true domain Ω_1 (stripes).

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