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$\mathcal{P}$-BIPARTITIONS OF MINOR HEREDITARY PROPERTIES

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Abstract

We prove that for any two minor hereditary properties $\mathcal{P}_1$ and $\mathcal{P}_2$, such that $\mathcal{P}_2$ covers $\mathcal{P}_1$, and for any graph $G \in \mathcal{P}_2$ there is a $\mathcal{P}_1$-bipartition of G. Some remarks on minimal reducible bounds are also included.

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1. Introduction and Notation

According to [3] we denote by $\mathcal{I}$ the class of all finite simple graphs. A *graph property* is a nonempty isomorphism-closed subclass of $\mathcal{I}$. We also say that a graph has the property $\mathcal{P}$ if $G \in \mathcal{P}$. For properties $\mathcal{P}_1, \mathcal{P}_2$ of graphs a vertex $(\mathcal{P}_1, \mathcal{P}_2)$-*partition* of a graph G is a partition (V_1, V_2) of $V(G)$ such that the subgraph $G[V_i]$ induced by the set V_i has the property $\mathcal{P}_i$ for each $i = 1, 2$. The class of all vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partitionable graphs is denoted by $\mathcal{P}_1 \circ \mathcal{P}_2$. If $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}$, then a $(\mathcal{P}_1, \mathcal{P}_2)$-partition (as in [4]) we call a $\mathcal{P}$-*bipartition*.

Let be given a graph $G \in \mathcal{I}$. A *contraction* of the graph G is a graph obtained from G by repeated contractions of edges, where *contraction of an edge* (v_1, v_2) of the graph G is obtained by deleting v_1 and v_2 and all incident edges from G and adding a new vertex u and all the edges required to satisfy the following condition $\mathrm{N}(u) = \mathrm{N}(v_1) \cup \mathrm{N}(v_2) \setminus \{v_1, v_2\}$.

A graph H obtained from G by deletions of vertices or edges, or contractions of edges is called a *minor* of G. So, the graph H is a minor of the graph G if H is a subgraph of G or can be obtained from a subgraph of G by contractions of edges. We express this relation between the graphs H and G by $H < G$.

A property $\mathcal{P}$ of graphs is called *minor hereditary* (*hereditary*) if it is closed under minors (subgraphs), i.e., if whenever $G \in \mathcal{P}$ and H is a minor (subgraph) of G, then also $H \in \mathcal{P}$.

Any minor hereditary property $\mathcal{P}$ can be uniquely determined by the set of *forbidden minors* which can be defined in the following way:

$$\boldsymbol{F_M}(\mathcal{P}) = \{G \in \mathcal{I} : G \notin \mathcal{P} \text{ but each minor } H \text{ of } G,\ H \neq G, \text{ belongs to } \mathcal{P}\}.$$

A property $\mathcal{P}$ is called *additive* if it is closed under disjoint union of graphs, i.e., if for each graph G all of whose connected components have a property $\mathcal{P}$ it follows that G has a property $\mathcal{P}$, too. It is easy to see that a minor hereditary property $\mathcal{P}$ is additive if and only if all minors $H \in \boldsymbol{F_M}(\mathcal{P})$ are connected.

Many well-known properties of graphs are both minor hereditary and additive. According to [2], [3] we list some of them to introduce the neccesary notions which will be used in the paper. It is convenient to work with an arbitrary nonnegative integer k.

$$\begin{aligned}
\mathcal{O} &= \{G \in \mathcal{I} : G \text{ is edgeless, i.e., } E(G) = \emptyset\},\\
\mathcal{D}_1 &= \{G \in \mathcal{I} : G \text{ is 1-degenerate, i.e., the minimum degree } \delta(H) \leq 1\\
&\qquad \text{for each } H \subseteq G\},\\
\mathcal{T}_k &= \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2}\\
&\qquad \text{or } K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\}, k \leq 3,\\
\mathcal{SP} &= \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_4\}.
\end{aligned}$$

We have $\mathcal{D}_1 = \mathcal{T}_1$ to be the class of all forests, $\mathcal{T}_2$ and $\mathcal{T}_3$ the class of all outerplanar and all planar graphs, respectively and $\mathcal{SP}$ the class of all series-parallel graphs.

For the properties given above we have:

$\boldsymbol{F_M}(\mathcal{O}) = \{K_2\}$,
$\boldsymbol{F_M}(\mathcal{D}_1) = \{K_3\}$,

$\boldsymbol{F_M}(\mathcal{T}_2) = \{K_4,\ K_{2,3}\}$,
$\boldsymbol{F_M}(\mathcal{T}_3) = \{K_5,\ K_{3,3}\}$,
$\boldsymbol{F_M}(\mathcal{SP}) = \{K_4\}$.

Let us define the next properties.

$\boldsymbol{F_M}(\mathcal{LF}) = \{K_3, K_{1,3}\}$,
$\boldsymbol{F_M}(\mathcal{S}) = \{K_4, K_{1,3} + K_1\}$.

All additive minor hereditary (hereditary) properties of graphs, partially ordered by a set-inclusion, form a lattice $\mathbf{T}^a$, ($\mathbb{L}^a$) with $\cap$ as a meet operation and $\mathcal{O}$ as the smallest element (see [2]).

All the above listed properties form in $\mathbf{T}^a$ the following chain:

$$\mathcal{O} \subset \mathcal{LF} \subset \mathcal{D}_1 \subset \mathcal{T}_2 \subset \mathcal{S} \subset \mathcal{SP} \subset \mathcal{T}_3.$$

2. $\mathcal{P}$-Bipartition Theorem

Definition. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two additive minor hereditary properties. We say that $\mathcal{P}_2$ *covers* $\mathcal{P}_1$ whenever for every graph $G_1 \in \boldsymbol{F_M}(\mathcal{P}_1)$ there exists a graph $G_2 \in \boldsymbol{F_M}(\mathcal{P}_2)$ such that $G_2 - v$ is a minor of G_1 for some vertex $v \in V(G_2)$.

Theorem 1. *If $\mathcal{P}_2$ covers $\mathcal{P}_1$, then the vertex set of a graph $G \in \mathcal{P}_2$ can be partitioned into two subsets such that each of them induces a subgraph of G belonging to $\mathcal{P}_1$.*

Proof. Let us consider a given graph $G \in \mathcal{P}_2$ with an arbitrarily choosen vertex v. It is sufficient to consider a case when G is connected. We define the subsets $U_k = \{u \in V(G) : d(v,u) = k\}$, where $d(u,v)$ is the length of the shortest path between v and u. Put $e = \max\{k : U_k \neq \emptyset\}$. Then $U_0, U_1, \ldots, U_e$ is a partition of $V(G)$ into $e+1$ pairwise disjoint subsets. Moreover, a subgraph induced by $U_0 = \{v\}$ belongs to $\mathcal{P}_1$. Now, let us assume to the contrary, that one of the subsets U_k, $k = 1, \ldots, e$, induces a subgraph of G, which is not in $\mathcal{P}_1$. Thus there is a minor H of $G[U_k]$ belonging to $\boldsymbol{F_M}(\mathcal{P}_1)$. Since the subgraph of G induced by $U' = \bigcup_{i=0}^{k-1} U_i$ is connected and every vertex of U_k is adjacent to a vertex of $U_{k-1} \subseteq U'$, then the graph $H + K_1$ is a minor of G. Since $\mathcal{P}_2$ covers $\mathcal{P}_1$, then $\boldsymbol{F_M}(\mathcal{P}_2)$ contains a graph H' such that $H' - u$ is a minor of H, for some $u \in V(H')$. Obviously, H' is a minor of $H + K_1$. Hence, since $H + K_1$ is a minor of G, then H' is a minor of G, contrary to $G \in \mathcal{P}_2$. Therefore, each of the subsets $U_i, i = 0, 1, \ldots, e$ induces a subgraph of G belonging to $\mathcal{P}_1$. Since vertices

$u \in U_i$ and $w \in U_j$, for $|i-j|>1$ are non-adjacent in G, then both of the sets $V_1 = \bigcup_{i=1}^{\lceil e/2 \rceil} U_{2i-1}$ and $V_2 = \bigcup_{i=0}^{\lfloor e/2 \rfloor} U_{2i}$ induce subgraphs of G belonging to $\mathcal{P}_1$, i.e., the partition (V_1, V_2) is the required $\mathcal{P}_1$-bipartition of $V(G)$. ■

From the theorem given above, a series of well-known results follows:

(a) $\mathcal{D}_1 \subset \mathcal{O}^2$,

(b) $\mathcal{T}_2 \subset \mathcal{LF}^2$
proved by Mihók [10], Broere and Mynhardt [5], Wang [13], and Goddard [8],

(c) $\mathcal{SP} \subset {\mathcal{D}_1}^2$
which is the result of Dirac [7],

(d) $\mathcal{T}_3 \subset {\mathcal{T}_2}^2$
proved by Broere and Mynhardt [5], Wang [13] and Poh [12].

The new conclusions can be drawn, too. For the class $\mathcal{S}$ defined by $\boldsymbol{F_M}(\mathcal{S}) = \{K_4, K_{1,3}+K_1\}$ we have:

(e) $\mathcal{S} \subset \mathcal{LF}^2$.

3. Minimal Reducible Bounds

An additive hereditary property $\mathcal{R}$ is called *reducible* in $\mathbb{L}^a$, if there exist additive hereditary properties $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$, and it is called *irreducible*, otherwise.

For a given property $\mathcal{P}$, a reducible property $\mathcal{R}$ is called *minimal reducible bound* for $\mathcal{P}$ if $\mathcal{P} \subset \mathcal{R}$ and there is no reducible property $\mathcal{R}' \subset \mathcal{R}$ satisfying $\mathcal{P} \subseteq \mathcal{R}'$. The set of all minimal reducible bounds for $\mathcal{P}$ will be denoted by $\boldsymbol{B}(\mathcal{P})$. The notion of minimal reducible bounds have been introduced in [11]. In this paper Mihók proved that the class $\mathcal{T}_2$ of outerplanar graphs has exactly two minimal reducible bounds, i.e., $\boldsymbol{B}(\mathcal{T}_2) = \{\mathcal{LF}^2, \mathcal{O}\circ\mathcal{D}_1\}$. A similar results for $\mathcal{SP}$ and $\mathcal{D}_2$ can be found in [1], namely, $\boldsymbol{B}(\mathcal{SP})=\boldsymbol{B}(\mathcal{D}_2) = \{\mathcal{O}\circ\mathcal{D}_1\}$.

By the transitivity and Mihók's proof (see [11]) we have the following minimal reducible bounds for the property $\mathcal{S} \supset \mathcal{T}_2$.

Theorem 2. $\boldsymbol{B}(\mathcal{S}) = \{\mathcal{LF}^2, \mathcal{O}\circ\mathcal{D}_1\}$.

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