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ON GENERALIZED LIST COLOURINGS OF GRAPHS

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Abstract

Vizing [15] and Erdös et al. [8] independently introduce the idea of considering list-colouring and k-choosability. In the both papers the choosability version of Brooks' theorem [4] was proved but the choosability version of Gallai's theorem [9] was proved independently by Thomassen [14] and by Kostochka et al. [11]. In [3] some extensions of these two basic theorems to $(\mathcal{P}, k)$-choosability have been proved.

In this paper we prove some extensions of the well-known bounds for the $\mathcal{P}$-chromatic number to the $(\mathcal{P}, k)$-choice number and then an extension of Brooks' theorem.

Keywords: hereditary property of graphs, list colouring, vertex partition number.

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1. Introduction and notation

All graphs considered in this paper are simple, i.e., finite, undirected, loopless and without multiple edges. The notation $(H \leq G)$ $H \subseteq G$ means that H is (an induced) subgraph of G. We say that G contains H whenever G contains (an induced) subgraph isomorphic to H.

For undefined concepts we refer the reader to [6, 10].

We denote by $\mathcal{I}$ the class of all finite simple graphs. A *graph property* is a nonempty isomorphism-closed subclass of $\mathcal{I}$. (We also say that a graph has the property $\mathcal{P}$ if $G \in \mathcal{P}$.) A property $\mathcal{P}$ of graphs is called (*induced*) *hereditary* if it is closed under (induced) subgraphs, i.e., if $(H \leq G)$ $H \subseteq G$ and $G \in \mathcal{P}$ imply $H \in \mathcal{P}$. A property $\mathcal{P}$ is called *additive* if it is closed under disjoint unions of graphs, i.e., if every graph has the property $\mathcal{P}$ provided all of its connected components have this property.

Many known properties of graphs are both hereditary and additive. According to [1], [2] we list some of them to introduce the necessary notions which will be used in the paper.

In what follows it is convenient to work with an arbitrary nonnegative integer k.

$$\begin{aligned}
\mathcal{O} &= \{G \in \mathcal{I} : G \text{ is edgeless, i.e., } E(G) = \emptyset\},\\
\mathcal{D}_k &= \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate, i.e., the minimum degree } \delta(H) \leq k\\
&\qquad \text{for each } H \subseteq G\},\\
\mathcal{T}_k &= \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2}\\
&\qquad \text{or } K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\},\\
\mathcal{SP} &= \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_4\}.
\end{aligned}$$

We have $\mathcal{D}_1 = \mathcal{T}_1$ to be the class of all forests, $\mathcal{T}_2$ and $\mathcal{T}_3$ the class of all outerplanar and all planar graphs, respectively and $\mathcal{SP}$ the class of all series-parallel graphs.

Let us denote by $(\mathbb{M})$, $\mathbb{L}$ and $(\mathbb{M}^a)$, $\mathbb{L}^a$ the set of all (induced) hereditary and additive (induced) hereditary properties, respectively. It can be proved that $(\mathbb{L}, \subseteq)$ is a proper sublattice of $(\mathbb{M}, \subseteq)$. For more details, see [1].

A hereditary property $\mathcal{P} \in \mathbb{M}$ is said to be nontrivial if $\mathcal{P} \neq \mathcal{I}$. Let $\delta(\mathcal{P}) = \min\{\delta(H) : H \in \mathcal{C}(\mathcal{P})\}$, where $\mathcal{C}(\mathcal{P}) = \{H \in \mathcal{I}\colon H \notin \mathcal{P}$ but $(H - v) \in \mathcal{P}$ for any $v \in V(H)\}$.

A $\mathcal{P}$-*partition* (*colouring*) of a graph G is a partition $(V_1, \ldots, V_n)$ of $V(G)$ such that the subgraph $G[V_i]$ induced by the set V_i has the property $\mathcal{P}$ for each $i = 1, \ldots, n$. If $(V_1, \ldots, V_n)$ is a $\mathcal{P}$-partition of a graph G, then the corresponding vertex colouring f is defined by $f(v) = i$ whenever $v \in V_i$, for $i = 1, \ldots, n$. The smallest integer n for which G has a $\mathcal{P}$-partition is called the $\mathcal{P}$-*chromatic* (or $\mathcal{P}$-*vertex-partition*) *number* of G and is denoted by $\chi_{\mathcal{P}}(G)$. The $\mathcal{O}$-chromatic number is the ordinary chromatic number. See [1], [2] for a survey and more details.

Let G be a graph and let $L(v)$ be a list of colours (say, positive integers) prescribed for the vertex v, and $\mathcal{P} \in \mathbb{M}$. A $(\mathcal{P}, L)$-*colouring* is a graph $\mathcal{P}$-colouring $f(v)$ with the additional requirement that for all $v \in V(G)$, $f(v) \in L(v)$. If G admits a $(\mathcal{P}, L)$-colouring, then G is said to be $(\mathcal{P}, L)$-*colourable.* The graph G is $(\mathcal{P}, k)$-*choosable* if it is $(\mathcal{P}, L)$-colourable for every list assignment L of G satisfying $|L(v)| = k$ for every $v \in V(G)$. The $\mathcal{P}$-*choice number* $\mathrm{ch}_{\mathcal{P}}(G)$ is the smallest natural number k such that G is $(\mathcal{P}, k)$-choosable.

For a nontrivial property $\mathcal{P} \in \mathbb{M}$ a graph G is said to be $(\mathcal{P}, L)$-*critical* if G has no $(\mathcal{P}, L)$-colouring but $G - v$ is $(\mathcal{P}, L)$-colourable for all $v \in V(G)$. A graph G is said to be (*vertex*) $(\mathcal{P}, k)$-*choice critical* if $\mathrm{ch}_{\mathcal{P}}(G) = k \geq 2$ but $\mathrm{ch}_{\mathcal{P}}(G - v) < k$ for all vertices v of G. According to the previous definitions, it follows immediately that if G is $(\mathcal{P}, k+1)$-choice critical, then G is $(\mathcal{P}, L)$-critical for some list assignment with $| L(v) | = k$ for all $v \in V(G)$.

The idea of considering $(\mathcal{O}, L)$-colouring and $(\mathcal{O}, k)$-choosability has been introduced independently by Vizing [15] and Erdös, Rubin and Taylor [8]. In both papers the choosability version of Brooks' theorem [4] was proved. The choosability version of Gallai's theorem [9] was proved independently by Thomassen [14] and by Kostochka et al. [11]. In [3] some extensions of these two basic theorems to $(\mathcal{P}, k)$-choosability, have been proved.

The aim of this paper is to prove some extensions of the well-known bounds for the $\mathcal{P}$-chromatic number [2], [3], [4] to the $(\mathcal{P}, k)$-choice number and an extension of Brooks' theorem for this number.

2. Results

Let us recall two results which we will use later.

Lemma 1 [3]**.** *Let* $\mathcal{P} \in \mathbb{M}$ *and* G *be* $(\mathcal{P}, L)$ *critical. Then* $d_G(v) \geq \delta(\mathcal{P}) \mid L(v) \mid$ *for any vertex* v *of* G.

Theorem 1 [3]**.** *Let* $\mathcal{P} \in \mathbb{M}^a$ *and* G *be a connected graph other than*

(i) *a complete graph of order* $n\delta(\mathcal{P}) + 1, n \geq 0$,

(ii) *a* $\delta(\mathcal{P})$*-regular graph belonging to* $\mathcal{C}(\mathcal{P})$,

(iii) *an odd cycle if* $\mathcal{P} = \mathcal{O}$.

Then

$$\mathrm{ch}_{\mathcal{P}}(G) \le \left\lceil \frac{\Delta(G)}{\delta(\mathcal{P})} \right\rceil .$$

Theorem 2. *Let $\mathcal{P} \in \mathbb{M}, \delta(\mathcal{P}) \ge 1$, and let f be a real valued function on graphs $G \in \mathcal{I}$ with properties*

(a) *$f(H) \le f(G)$, for any induced subgraph H of G,*

(b) *$f(G) \ge \delta(G)$, for each graph.*

Then

$$\mathrm{ch}_{\mathcal{P}}(G) \le \left\lfloor \frac{f(G)}{\delta(\mathcal{P})} \right\rfloor + 1.$$

Proof. Obviously, the theorem holds for G with $\mathrm{ch}_{\mathcal{P}}(G) = 1$. Let $\mathrm{ch}_{\mathcal{P}}(G) = t + 1 \ge 2$, and let H be a $(\mathcal{P}, t+1)$-choice critical induced subgraph of G. Thus there exists a list assignement with $|L(v)| = t$ for all $v \in V(G)$ and H is $(\mathcal{P}, L)$-critical. Lemma 1 implies $\delta(H) \ge t\delta(\mathcal{P})$, and (a) yields that $f(H) \le f(G)$. Thus, by (b) we have $t\delta(\mathcal{P}) \le \delta(H) \le f(H) \le f(G)$. Hence,

$$\mathrm{ch}_{\mathcal{P}}(G) = t + 1 \le \left\lfloor \frac{f(G)}{\delta(\mathcal{P})} \right\rfloor + 1.$$

Corollary 1. *Let $\rho(G) = \max\{\delta(H) \colon H \le G\}$. Then*

$$\mathrm{ch}_{\mathcal{P}}(G) \le \left\lfloor \frac{\rho(G)}{\delta(\mathcal{P})} \right\rfloor + 1.$$

Proof. It easy to see that $\rho(G)$ satisfies (a) and (b) of the above theorem.

Corollary 2. *Let $\mathcal{P} = \mathcal{D}_r$ and $G \in \mathcal{D}_k$, $r \le k$. Then*

$$\mathrm{ch}_{\mathcal{P}}(G) \le \left\lfloor \frac{k}{r+1} \right\rfloor + 1.$$

Proof. The statement follows by the previous corollary because $\rho(G) \le k$ for every k-degenerate graph G and $\delta(\mathcal{D}_r) = r + 1$.

Corollary 3. *Let $G \in \mathcal{T}_3$. Then $\mathrm{ch}_{\mathcal{D}_1}(G) \le 3$.*

Proof. The statement follows by the previous corollary because any planar graph is 5-degenerate.

Corollary 3 implies the well-known result of Chartrand and Kronk [5]: *The vertex-arboricity of any planar graph is at most* 3.

Similarly as above, we have a corresponding result for series-parallel and outerplanar graphs which are 2-degenerate. This Corollary implies Dirac's [7] result: $\mathcal{SP} \subset \mathcal{D}_1^2$.

Corollary 4. *Let $G \in \mathcal{SP}$. Then* $\mathrm{ch}_{\mathcal{D}_1}(G) \leq 2$.

In [12] Mihók obtained an extension of Brooks' theorem. From the previous results and the following Mihók's lemma we derive a choosability version of his result.

Lemma 2 [12]**.** *Let G be a connected graph with $\delta(G) \geq k \geq 1$, other than K_{k+1}. Then G contains as a subgraph every tree on $k+2$ vertices, except for $T = K_{1,k+1}$ if G is k-regular.*

Theorem 3. *Let $\mathcal{P} \in \mathbb{L}^a$ with $\delta(\mathcal{P}) \geq 1$. If a connected graph G does not contain a given tree T on $k+2$ vertices, $k \geq 3$, as a subgraph, then G is $(\mathcal{P}, \lceil k/\delta(\mathcal{P}) \rceil)$-choosable unless $G = K_{k+1}$.*

Proof. Assume that G is a connected graph $\neq K_{k+1}$ and G does not contain a given tree T on $k+2$ vertices. If $T = K_{1,k+1}$, then by Theorem 1, G is $(\mathcal{P}, \lceil k/\delta(\mathcal{P}) \rceil)$-choosable. If $T \neq K_{1,k+1}$, then by Lemma 2, G must be $(k-1)$-degenerate, otherwise G contains T. Thus by Corollary 1, G is $(\mathcal{P}, \lceil k/\delta(\mathcal{P}) \rceil)$-choosable, too.

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