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THE ORDER OF UNIQUELY PARTITIONABLE GRAPHS

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Abstract

Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be properties of graphs. A $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-*partition* of a graph G is a partition $\{V_1, \ldots, V_n\}$ of $V(G)$ such that, for each $i = 1, \ldots, n$, the subgraph of G induced by V_i has property $\mathcal{P}_i$. If a graph G has a unique $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition we say it is *uniquely* $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-*partitionable.* We establish best lower bounds for the order of uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graphs, for various choices of $\mathcal{P}_1, \ldots, \mathcal{P}_n$.

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1. Introduction and Notation

All graphs considered in this paper are finite, undirected, loopless and without multiple edges. For undefined concepts we refer the reader to [5] and [2].

We denote the set of all mutually non-isomorphic graphs by $\mathcal{I}$. If $\mathcal{P}$ is a non-empty proper subset of $\mathcal{I}$, then $\mathcal{P}$ will also denote the property that a graph is a member of the set $\mathcal{P}$. We shall use the terms *set of graphs* and *property of graphs* interchangeably.

A property $\mathcal{P}$ is called *additive* if for each graph G all of whose components have property $\mathcal{P}$ it follows that $G \in \mathcal{P}$ too. A property $\mathcal{P}$ is *hereditary* whenever it is closed with respect to the relation $\subseteq$ to be a *subgraph*.

In the sequel we shall concentrate on the following concrete hereditary properties (we used the notation of [2, 9]):

$$\begin{aligned}
\mathcal{O} &= \{G \in \mathcal{I} : G \text{ is totally disconnected}\},\\
\mathcal{O}_k &= \{G \in \mathcal{I} : \text{ each component of } G \text{ has at most } k+1 \text{ vertices}\},\\
\mathcal{S}_k &= \{G \in \mathcal{I} : \Delta(G) \leq k\},\\
\mathcal{Q}_k &= \{G \in \mathcal{I} : \text{ the length of the longest path in } G \text{ does not exceed } k\},\\
\mathcal{D}_k &= \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate}\},\\
\mathcal{I}_k &= \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\}.
\end{aligned}$$

Let $\mathcal{P}$ be a hereditary property, $\mathcal{P} \neq \mathcal{I}$. Then there is a nonnegative integer $c(\mathcal{P})$ such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \notin \mathcal{P}$, called the *completeness* of $\mathcal{P}$. Obviously

$$c(\mathcal{O}_k) = c(\mathcal{S}_k) = c(\mathcal{Q}_k) = c(\mathcal{D}_k) = c(\mathcal{I}_k) = k$$

and $c(\mathcal{P}) = 0$ if and only if $\mathcal{P} = \mathcal{O}$.

It is easy to verify that $\mathcal{O}_k \subseteq \mathcal{S}_k \subseteq \mathcal{D}_k \subseteq \mathcal{I}_k$ and $\mathcal{O}_k \subseteq \mathcal{Q}_k \subseteq \mathcal{D}_k$.

If $\mathcal{P} \subseteq \mathcal{I}$ is a hereditary property, we define the set of *minimal forbidden subgraphs* of $\mathcal{P}$ as follows:

$$\boldsymbol{F}(\mathcal{P}) = \{G \in \mathcal{I} : G \notin \mathcal{P} \text{ but each proper subgraph of } G \text{ belongs to } \mathcal{P}\}.$$

Lemma 1.1. *Let $\mathcal{P}$ be a hereditary property. Then $G \in \mathcal{P}$ if and only if no subgraph of G is in $\boldsymbol{F}(\mathcal{P})$.* ∎

Thus any hereditary property is uniquely determined by its set of minimal forbidden subgraphs.

An alternative way is to characterize $\mathcal{P}$ by the set of graphs containing all the graphs in $\mathcal{P}$ as subgraphs. To be more accurate, let us define the set of $\mathcal{P}$-*maximal graphs* by

$$\boldsymbol{M}(\mathcal{P}) = \{G \in \mathcal{P} : G + e \notin \mathcal{P} \text{ for each } e \in E(\overline{G})\}$$

and the set of $\mathcal{P}$-*maximal graphs of order* n by

$$\boldsymbol{M}(n, \mathcal{P}) = \{G \in \mathcal{P} : |V(G)| = n \text{ and } G + e \notin \mathcal{P} \text{ for each } e \in E(\overline{G})\}.$$

We say that a graph G is the *join* of n graphs $G_1, \ldots, G_n$ and write $G = G_1 + \cdots + G_n$ if

$$\begin{aligned} V(G) &= \bigcup_{i=1}^{n} V(G_i) \text{ and} \\ E(G) &= \{xy : xy \in E(G_i) \text{ for some } i \text{ or } x \in V(G_i) \text{ and } y \in V(G_j);\ i \neq j\}. \end{aligned}$$

If a graph G is a join of non-empty graphs, we say that G is *decomposable*; otherwise, G is *indecomposable*.

If $\mathcal{P}$ is a hereditary property then, clearly, the only $\mathcal{P}$-maximal graphs of order less than $c(\mathcal{P})+2$ are complete graphs and thus they are decomposable or trivial. The next two results, concerning indecomposable nontrivial $\mathcal{P}$-maximal graphs of smallest possible order for certain properties $\mathcal{P}$, are proved in [4].

Proposition 1.2. *If $\mathcal{P}$ is any additive, hereditary property with $c(\mathcal{P}) = k \geq 1$ and $\boldsymbol{F}(\mathcal{P})$ contains some tree of order $k+2$, then the graph $K_{k+1} \cup K_1$ is an indecomposable $\mathcal{P}$-maximal graph of order $c(\mathcal{P}) + 2$.* ■

Properties that satisfy the conditions of Proposition 1.2 are, for example, $\mathcal{O}_k$, $\mathcal{S}_k$ and $\mathcal{Q}_k$. However, $\boldsymbol{F}(\mathcal{I}_k)$ contains no trees, and for $\mathcal{I}_k$ we have

Theorem 1.3. *If G is an indecomposable nontrivial $\mathcal{I}_k$-maximal graph, then $|V(G)| \geq 2k+3$, with equality only if $G = \overline{C_{2k+3}}$.* ■

Let n be a positive integer and let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be properties of graphs. A $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-*partition of a graph* G is a partition $\{V_1, \ldots, V_n\}$ of $V(G)$ such that, for each $i = 1, \ldots, n$, the induced subgraph $G[V_i]$ has property $\mathcal{P}_i$.

If $\mathcal{P}_1 = \cdots = \mathcal{P}_n$, we shall call a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition a $(\mathcal{P}^n)$-partition.

The property $\mathcal{R} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$ is defined as the set of all graphs having a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition. If $\mathcal{P}_1 = \cdots = \mathcal{P}_n = \mathcal{P}$, we write $\mathcal{R} = \mathcal{P}^n$.

A graph G is said to be *uniquely* $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-*partitionable* if and only if G has a unique $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition (permutation of partition sets are allowed). Note that, if G is uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable and

$\{V_1, \ldots, V_n\}$ is the unique $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of G, then $V_i \neq \emptyset$ for $i = 1, \ldots, n$. We denote the class of all uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graphs by $\boldsymbol{U}(\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n)$, and if $\mathcal{P}_1 = \cdots = \mathcal{P}_n$, we also write it as $\boldsymbol{U}(\mathcal{P}^n)$ (see [4, 7, 8]).

We shall show that, for certain properties, joins of indecomposable maximal graphs yield uniquely partitionable graphs, and then we shall use Proposition 1.2 and Theorem 1.3 to establish best lower bounds for the order of uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graphs, for various properties $\mathcal{P}_1, \ldots, \mathcal{P}_n$.

2. Maximal Uniquely Partitionable Graphs

We say that G is a *maximal uniquely* $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$*-partitionable graph* if $G \in \boldsymbol{U}(\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n)$ but $G + e \notin \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$ for any $e \in E(\overline{G})$).

Proposition 2.1. *If G is a maximal uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graph, then*

$G = G_1 + \cdots + G_n$ where $G_i \in \boldsymbol{M}(\mathcal{P}_i)$ and $|V(G_i)| \geq c(\mathcal{P}_i) + 1$ for $i = 1, \ldots, n$.

Proof. Let $\{V_1, \ldots, V_n\}$ be the $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of G. Put $G_i = G[V_i]$ for $i = 1, \ldots, n$. If a vertex x of G_i is non-adjacent to a vertex y of G_j; $i \neq j$, then $\{V_1, \ldots, V_n\}$ is also a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of $G + xy$, and hence $G + xy \in \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$, contradicting the maximality of G. Thus $G = G_1 + \cdots + G_n$. If $G_i \notin \boldsymbol{M}(\mathcal{P}_i)$ for some i, then there is an edge e in $\overline{G_i}$ such that $G_i + e \in \mathcal{P}_i$, and then $G + e \in \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$, again contradicting the maximality of G. If $|V(G_i)| \leq c(\mathcal{P}_i)$, then another $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of G can be obtained from $\{V_1, \ldots, V_n\}$ by removing a vertex from any V_j; $j \neq i$, and adding it to V_i. ■

The converse of Proposition 2.1 is not true, i.e. if $G_i \in \boldsymbol{M}(\mathcal{P}_i)$; $i = 1, \ldots, n$ and $|V(G_i)| \leq c(\mathcal{P}_i) + 1$, then $G_1 + \cdots + G_n$ need not be uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable. However, we shall show that, if at least $n - 1$ of the graphs G_i are indecomposable and the properties satisfy certain requirements, then $G_1 + \cdots + G_n$ is uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable. First, we need a lemma.

Lemma 2.2. *Let $\mathcal{P}$ be any hereditary property and suppose G is an indecomposable $\mathcal{P}$-maximal graph. If V_1 and V_2 are non-empty subsets of $V(G)$ such that $V(G) = V_1 \cup V_2$, then $G[V_1] + G[V_2]$ is not in $\mathcal{P}$.*

Proof. Since G is indecomposable there is a vertex v_1 in V_1 and a vertex v_2 in V_2 such that v_1v_2 is not in $E(G)$. Therefore, since G is $\mathcal{P}$-maximal,

$G + v_1v_2$ is not in $\mathcal{P}$. But $G + v_1v_2$ is a subgraph of $G[V_1] + G[V_2]$ and therefore, since $\mathcal{P}$ is hereditary, $G[V_1] + G[V_2]$ is also not in $\mathcal{P}$. ■

Theorem 2.3. *If $G_1, \ldots, G_n$ are graphs such that G_i is an $\mathcal{I}_{k_i}$-maximal graph of order at least $k_i + 1$ for $i = 1, \ldots, n$ and G_i is indecomposable for $i = 1, \ldots, n-1$, then the join $G_1 + \cdots + G_n$ is a maximal uniquely $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partitionable graph.*

Proof. Let $G = G_1 + \cdots + G_n$ and suppose $\{V_1, \ldots, V_n\}$ is an $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partition of $V(G)$ different from $\{V(G_1), \ldots, V(G_n\}$. Then at least one of the V_i, say V_n, contains vertices from at least two different G_i's. Let

$$W_i = V(G_i) \cap V_n \quad \text{for } i = 1, \ldots, n.$$

Suppose $W_i \neq \emptyset$ for $i = 1, \ldots, r$; $1 \leq r \leq n-1$ and $W_i = \emptyset$ if $r < i < n$. Then $V_n = (\cup_{i=1}^r W_i) \cup W_n$ and hence

$$G[V_n] = G[W_1] + \cdots + G[W_r] + G[W_n].$$

(Note that W_n may be empty.)

It follows from Lemma 2.2 that $(G_i - W_i) + W_i$ is not in $\mathcal{I}_{k_i}$, and hence

$$\omega(G_i - W_i) + \omega(G[W_i]) \geq k_i + 2$$

for $i = 1, \ldots, r, n$. Now

$$\begin{aligned}
\omega(G - V_n) &= \omega[(G_1 - W_1) + \cdots + (G_r - W_r) \\
&\quad + G_{r+1} + \cdots + G_{n-1} + (G_n - W_n)] \\
&= \sum_{i=1}^{r} \omega(G_i - W_i) + \sum_{i=r+1}^{n-1} \omega(G_i) + \omega(G_n - W_n) \\
&\geq \sum_{i=1}^{r} (k_i + 2 - \omega(G[W_i])) + \sum_{i=r+1}^{n-1} (k_i + 1) + k_n + 1 - \omega(G[W_n]) \\
&= \sum_{i=1}^{n} (k_i + 1) - \omega(G[W_1] + \cdots + G[W_r] + G[W_n]) + r \\
&\geq \sum_{i=1}^{n} (k_i + 1) - (k_n + 1) + r \quad (\text{since } \omega(G[V_n]) \leq k_n + 1) \\
&= \sum_{i=1}^{n-1} (k_i + 1) + r.
\end{aligned}$$

However, $G - V_n$ is a subgraph of $G[V_1] + \cdots + G[V_{n-1}]$ and hence

$$\begin{aligned}\omega(G - V_n) &\leq \sum_{i=1}^{n-1} \omega(G[V_i]) \\ &\leq \sum_{i=1}^{n-1} (k_i + 1).\end{aligned}$$

This contradiction implies that $\{V(G_1), \ldots, V(G_n)\}$ is the only $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partition of G. ■

We call a graph invariant γ *plus-preserving* if

$$\gamma(G_1 + G_1) = \gamma(G_1) + \gamma(G_2) \text{ for all } G_1, G_2 \in \mathcal{I}.$$

If γ is any plus-preserving invariant and $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are hereditary, additive properties such that

$$\mathcal{P}_i = \{G \in \mathcal{I} : \gamma(G) \leq k_i\};\ i = 1, \ldots, n$$

then Theorem 2.3 also holds if $\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n}$ are replaced by $\mathcal{P}_1, \ldots, \mathcal{P}_n$.

The following theorem is proved in [1].

Theorem 2.4. *If $G_1, \ldots, G_n$ are $\mathcal{Q}_k$-maximal graphs of order at least $k+1$ each, and at least $n-1$ of them have no universal vertices, then $G_1 + \cdots + G_n$ is a maximal uniquely $(\mathcal{Q}_k^n)$-partitionable graph.* ■

The following result is proved in [4].

Lemma 2.5. *A nontrivial $\mathcal{Q}_k$-maximal graph is indecomposable if and only if it contains no universal vertices.* ■

As a corollary of the last two results, we have

Theorem 2.6. *A graph G is a maximal uniquely $(\mathcal{Q}_k^n)$-partitionable graph if and only if $G = G_1 + \cdots + G_n$, where $G_1, \ldots, G_n$ are $\mathcal{Q}_k$-maximal graphs of order at least $k+1$, and at least $n-1$ of the G_i are indecomposable.*

Proof. The "if"-part follows directly from Theorem 2.4 and Lemma 2.5. For the "only if"-part, suppose G is a maximal uniquely $(\mathcal{Q}_k^n)$-partitionable graph. Then, by Proposition 2.1, $G = G_1 + \cdots + G_n$, where G_i is a $\mathcal{Q}_k$-maximal graph of order at least $k+1$; $i = 1, \ldots, n$ and $\{V(G_1), \ldots, V(G_n)\}$

is the only $(\mathcal{Q}_k^n)$-partition of G. Now suppose G_1 and G_2 are both decomposable. Then, by Lemma 2.5, G_i has a universal vertex x_i, $i = 1, 2$. But then another $(\mathcal{Q}_k^n)$-partition of G can be obtained from $\{V(G_1), \ldots, V(G_n)\}$ by interchanging x_1 and x_2. ■

3. The Order of Uniquely Partitionable Graphs

Theorem 3.1. *Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$, $n \geq 2$, be hereditary properties of graphs and suppose G is a uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graph. Then*

$$| V(G) | \geq \sum_{i=1}^{n} (c(\mathcal{P}_i) + 2) - 1.$$

Proof. Let $\{V_1, \ldots, V_n\}$ be the unique $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of $V(G)$. Suppose that $|V_i| \leq c(\mathcal{P}_i)$ for some i. Then, by the definition of $c(\mathcal{P}_i)$, we have $G[V_i \cup \{x\}] \in \mathcal{P}_i$ for any $x \in V(G) - V_i$. Thus another $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of $V(G)$ can be obtained from $\{V_1, \ldots, V_n\}$ by removing a vertex from V_j for some $j \neq i$ and adding this vertex to V_i. This contradiction implies that $|V_i| \geq c(\mathcal{P}_i) + 1$ for $i = 1, \ldots, n$.

Now suppose that $|V_i| = c(\mathcal{P}_i) + 1$ and $|V_j| = c(\mathcal{P}_j) + 1$ for some i and j with $i \neq j$. Then another $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of $V(G)$ can be obtained from $\{V_1, \ldots, V_n\}$ by interchanging a vertex of V_i with a vertex of V_j. Thus $|V_i| \geq c(\mathcal{P}_i) + 2$ for every $i \in \{1, \ldots, n\}$ except perhaps for one i. The result follows. ■

Our next theorem will show that the bound of Theorem 3.1 is the best possible for certain properties. First, we need a lemma.

Lemma 3.2. *If a graph G has a subgraph $A = K_k$, and two distinct vertices v_1 and v_2 of $V(G) - A$ are adjacent to distinct vertices a_1 and a_2 of A respectively, then G contains every tree of order $k + 2$.*

Proof. Let T be any tree of order $k + 2$ and let x and y be any two end vertices of T. Then $T - \{x, y\}$ is a subgraph of A, and the result follows. ■

Theorem 3.3. *Suppose $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are additive, hereditary properties such that $\boldsymbol{F}(\mathcal{P}_i)$ contains some tree T_i of order $c(\mathcal{P}_i) + 2$ for each $i = 1, \ldots, n$. Then there exists a uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graph G with*

$$|V(G)| = \sum_{i=1}^{n} (c(\mathcal{P}_i) + 2) - 1.$$

Proof. Let $c(\mathcal{P}_i) = k_i$ and put

$$\begin{aligned} G_i &= K_{k_i+1} \cup K_1 \text{ for } i = 1, \ldots, n-1, \\ G_n &= K_{k_n+1} \\ \text{and } G &= G_1 + \cdots + G_n. \end{aligned}$$

Let x_i denote the isolated vertex of G_i; $i = 1, \ldots, n-1$ and let $X = \{x_1, \ldots, x_{n-1}\}$.

Clearly, $\{V(G_1), \ldots, V(G_n)\}$ is a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of G. Now suppose $\{V_1, \ldots, V_n\}$ is another $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of G. Note that, if $|V_i \cap X| = \emptyset$ for some i, then $G[V_i]$ is a complete graph and then, since $c(\mathcal{P}_i) = k_i$, it follows that $|V_i| \leq k_i + 1$.

Now suppose that

$$|V_i| = k_i + s_i; \quad i = 1, \ldots, n.$$

If $s_i \geq 2$ then $|V_i \cap X| \geq 1$, otherwise $G[V_i]$ would contain a K_{k_i+2}.

If $s_i \geq 3$, then $|V_i \cap X| \geq s_i + 1$, otherwise the vertices of V_i that are not in X would induce a K_{k_i} in G and two of the vertices of $|V_i \cap X|$ will be adjacent to different vertices of this K_{k_i}, so that, by Lemma 3.2, T_i will be a subgraph of $V[G_i]$.

Suppose

$$\begin{aligned} s_i &\geq 3 \text{ for } i = 1, \ldots, l, \\ s_i &= 2 \text{ for } i = l+1, \ldots, m, \\ s_i &\leq 1 \text{ for } i = m+1, \ldots, n. \end{aligned}$$

Then

$$|V_i \cap X| \geq s_i + 1 \text{ for } i = 1, \ldots, l$$

and hence, since $|X| = n-1$, at least $1 + \sum_{i=1}^{l} s_i$ of the V_i contain at least one element of X. Thus

$$n - m \geq 1 + \sum_{i=1}^{l} s_i$$

and therefore

$$\begin{aligned} \sum_{i=1}^{n} |V_i| &\leq \sum_{i=1}^{l} (k_i + s_i) + \sum_{i=l+1}^{m} (k_i + 2) + \sum_{i=m+1}^{n} (k_i + 1) \\ &= \sum_{i=1}^{n} (k_i + 2) + \sum_{i=1}^{l} s_i - 2l - (n - m) \end{aligned}$$

$$\leq \sum_{i=1}^{n}(k_i+2)+n-m-1-2l-(n-m)$$
$$= \sum_{i=1}^{n}(k_i+2)-1-2l.$$

Since $|V(G)| = \sum_{i=1}^{n}(k_i+2)-1$, this proves that $l = 0$ so that $s_i \leq 2$ for $i = 1, \ldots, n$.

Suppose $|V_i| \leq k_i + 1$ for some index i. Then we must have that $|V_i| = k_i+1$ for only one index and $|V_i| = k_i+2$ for the remaining indices. It follows that $V_i \cap X = \emptyset$ for only one index and that $|V_i \cap X| = 1$ for $n-1$ of the indices. If, for one of these $n-1$ indices there are vertices from two different G_j's in V_i, then it follows from Lemma 3.2 that T_i is a subgraph of $G[V_i]$. Thus each V_i is contained in some $V(G_j)$, and hence $\{V(G_1), \ldots, V(G_n)\}$ is the only $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of G.

Hence we may assume that $|V_i| = k_i + 2$ for each i. But then

$$\sum_{i=1}^{n}|V_i| = \sum_{i=1}^{n}(k_i+2),$$

a contradiction. ■

Theorem 3.3 can be applied, for example, if $\mathcal{P}_i = \mathcal{O}_{k_i}$, $\mathcal{Q}_{k_i}$ or $\mathcal{S}_{k_i}$; $i = 1, \ldots, n$. However, if $\mathcal{P}_i = \mathcal{I}_{k_i}$ with $k_i \geq 1$ for $i = 1, \ldots, n$, the bound of Theorem 3.1 is not best possible. In order to establish the best bound in this case, we need some definitions and lemmas.

An *elementary homomorphism* of a graph is an identification of two non-adjacent vertices of the graph. Following [6], we call a graph G *k-replete* if $\omega(G) = k$ and $\omega(\phi(G)) > k$ for every elementary homomorpism ϕ of G. The following result is proved in [6].

Lemma 3.4. *If G is a $(k+1)$-replete graph without universal vertices, then $|V(G)| \geq 2k+3$, with equality only if $G = \overline{C_{2k+3}}$.* ■

Lemma 3.5. *If G is uniquely $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partitionable and ϕ is an elementary homomorphism of G such that $\phi(G) \in \mathcal{I}_{k_1} \circ \cdots \circ \mathcal{I}_{k_n}$, then $\phi(G)$ is also uniquely $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partitionable.*

Proof. Let $\{V_1, \ldots, V_n\}$ be any $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partition of $\phi(G)$. Let x and y be the two non-adjacent vertices of G that are identified by ϕ. Suppose $\phi(x) = \phi(y) \in V_1$. Then $\omega(\phi^{-1}(V_1)) \leq \omega(V_1)$, and hence $\{\phi^{-1}(V_1), \ldots, V_n\}$ is an $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partition of G. Since G has a unique $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partition, it follows that $\phi(G)$ also has a unique $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partition. ■

Theorem 3.6 *If G is a uniquely $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partitionable graph and $k_n \geq k_i$ for $i = 1, \ldots, n$, then*

$$|V(G)| \geq \sum_{i=1}^{n-1} (2k_i + 3) + k_n + 1,$$

with equality only if

$$G = \overline{C_{2k_1+3}} + \cdots + \overline{C_{2k_{n-1}+3}} + K_{k_n+1}.$$

Proof. First, we note that, since $\overline{C_{2k_i+3}}$ is an indecomposable $\mathcal{I}_{k_i}$-maximal graph, it follows from Theorem 2.3 that the graph $\overline{C_{2k_1+3}} + \cdots + \overline{C_{2k_{n-1}+3}} + K_{k_n+1}$ is indeed a maximal uniquely $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partitionable graph.

Now let G be a maximal uniquely $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partitionable graph of smallest possible order, and let $\{V_1, \ldots, V_n\}$ be the unique $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partition of G. Put

$$G = G_1 + \ldots + G_n.$$

Now suppose G_i is not $(k_i + 1)$-replete for some $i \in \{1, \ldots, n\}$. Then there is an elementary homomorphism ϕ of G_i such that $\omega(\phi(G_i)) \leq k_i + 1$. Now ϕ can also be regarded as an elementary homomorphism of G, and $\{V(G_1), \ldots, V(G_{i-1}), V(\phi(G_i), V(G_{i+1}), \ldots V(G_n)\}$ is an $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partition of $\phi(G)$. Thus $\phi(G) \in \mathcal{I}_{k_1} \circ \cdots \circ \mathcal{I}_{k_n}$ and hence, by Lemma 3.5, $\phi(G)$ is a uniquely $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partitionable graph of order less than $|V(G)|$. This contradiction proves that each G_i is a replete graph.

Now suppose that G_i as well as G_j have a universal vertex, for $i \neq j$. Then these two vertices can be interchanged in such a way that we obtain an $(\mathcal{I}_{k_1}, \ldots, \mathcal{I}_{k_n})$-partition of G different from $\{V_1, \ldots, V_n\}$. This proves that at most one of the G_i contains a universal vertex, and the result now follows from Proposition 2.1 and Lemma 3.4. ■

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