

ON SOME VARIATIONS OF EXTREMAL GRAPH PROBLEMS

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Abstract

A set \mathcal{P} of graphs is termed *hereditary property* if and only if it contains all subgraphs of any graph G belonging to \mathcal{P} . A graph is said to be *maximal* with respect to a hereditary property \mathcal{P} (shortly *\mathcal{P} -maximal*) whenever it belongs to \mathcal{P} and none of its proper supergraphs of the same order has the property \mathcal{P} . A graph is *\mathcal{P} -extremal* if it has a the maximum number of edges among all \mathcal{P} -maximal graphs of given order. The number of its edges is denoted by $\text{ex}(n, \mathcal{P})$. If the number of edges of a \mathcal{P} -maximal graph is minimum, then the graph is called *\mathcal{P} -saturated* and its number of edges is denoted by $\text{sat}(n, \mathcal{P})$.

In this paper, we consider two famous problems of extremal graph theory. We shall translate them into the language of \mathcal{P} -maximal graphs and utilize the properties of the lattice of all hereditary properties in order to establish some general bounds and particular results. Particularly, we shall investigate the behaviour of $\text{sat}(n, \mathcal{P})$ and $\text{ex}(n, \mathcal{P})$ in some interesting intervals of the mentioned lattice.

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1. HEREDITARY PROPERTIES OF GRAPHS

All graphs considered in this paper are ordinary and finite. The nature of our considerations allows us to restrict our attention to the set \mathcal{I} of all mutually nonisomorphic graphs. For the sake of brevity, we shall say "a graph G contains a subgraph H " instead of "a graph G contains a subgraph isomorphic to a graph H ".

A nonempty subset \mathcal{P} of \mathcal{I} is called *hereditary property*, whenever it is closed under subgraphs. In other words, if G is any graph from \mathcal{P} and H is its subgraph, then H also belongs to \mathcal{P} . A hereditary property is named *additive*, whenever it is closed under disjoint union of graphs.

In what follows we shall deal with the following examples of hereditary properties:

$$\begin{aligned} \mathcal{O} &= \{G \in \mathcal{I} : G \text{ is totally disconnected}\}, \\ \mathcal{O}_k &= \{G \in \mathcal{I} : \text{each component of } G \text{ has at most } k+1 \text{ vertices}\}, \\ \mathcal{D}_k &= \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate}\}, \\ \mathcal{T}_k &= \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2} \text{ or} \\ &\quad K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\}, \\ \mathcal{I}_k &= \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\}. \end{aligned}$$

Any hereditary property \mathcal{P} can be uniquely determined either by the set of graphs not appearing in \mathcal{P} (even as a subgraphs) or by the set of maximal admissible graphs (for details see e.g. [1]). More precisely, let us define the sets $\mathbf{F}(\mathcal{P})$ of *minimal forbidden subgraphs* and $\mathbf{M}(\mathcal{P})$ of \mathcal{P} -maximal graphs in the following manner:

$$\begin{aligned} \mathbf{F}(\mathcal{P}) &= \{F \in \mathcal{I} \setminus \mathcal{P} : \text{any proper subgraph } F^* \text{ of } F \text{ belongs to } \mathcal{P}\}, \\ \mathbf{M}(\mathcal{P}) &= \bigcup_{n=1}^{\infty} \mathbf{M}(n, \mathcal{P}), \\ \mathbf{M}(n, \mathcal{P}) &= \{G \in \mathcal{P} : |V(G)| = n \text{ and } G + e \notin \mathcal{P} \text{ for any edge } e \in E(\overline{G})\}. \end{aligned}$$

In the next sections, we shall often need the following useful lemmas.

Lemma 1. *Let $\mathcal{P}_1, \mathcal{P}_2$ be any hereditary properties. Then the following statements are mutually equivalent:*

1. $\mathcal{P}_1 \subseteq \mathcal{P}_2$;
2. for each $H \in \mathbf{F}(\mathcal{P}_2)$ there exists $H' \in \mathbf{F}(\mathcal{P}_1)$ such that $H' \subseteq H$;
3. for any positive integer n and an arbitrary $G \in \mathbf{M}(n, \mathcal{P}_1)$ there is $G' \in \mathbf{M}(n, \mathcal{P}_2)$ such that $G \subseteq G'$.

Proof. (1) \Rightarrow (2). Let $H \in \mathbf{F}(\mathcal{P}_2)$. Then $H \notin \mathcal{P}_1$, and clearly H is not a subgraph of any $G \in \mathcal{P}_1$. Hence, there exists $H' \in \mathbf{F}(\mathcal{P}_1)$ such that $H' \subseteq H$.

(2) \Rightarrow (3). If $G \in \mathbf{M}(n, \mathcal{P}_1)$, then G does not possess any $H' \in \mathbf{F}(\mathcal{P}_1)$. Thus G does not contain any $H \in \mathbf{F}(\mathcal{P}_2)$ and therefore either $G \in \mathbf{M}(n, \mathcal{P}_2)$ or there exists $G' \in \mathbf{M}(n, \mathcal{P}_2)$ such that $G \subseteq G'$.

(3) \Rightarrow (1). This implication follows immediately from the definitions. ■

Lemma 2. *Let \mathcal{P}_1 and \mathcal{P}_2 be any hereditary properties of graphs. If $\mathcal{P}_1 \subseteq \mathcal{P}_2$, $G \in \mathbf{M}(n, \mathcal{P}_2)$ and $G \in \mathcal{P}_1$, then G belongs to $\mathbf{M}(n, \mathcal{P}_1)$.*

Proof. If $G \in \mathbf{M}(n, \mathcal{P}_2)$, then for each edge e of the complement of G we have $G + e \notin \mathcal{P}_2$. Hence, $G + e \notin \mathcal{P}_1$ for any edge $e \in E(\overline{G})$. Then since $G \in \mathcal{P}_1$, we get $G \in \mathbf{M}(n, \mathcal{P}_1)$. ■

It is not so difficult to see that for any hereditary property \mathcal{P} , which is distinct from \mathcal{I} , there exists the number $c(\mathcal{P})$ (called the *completeness* of \mathcal{P}) defined as follows: $c(\mathcal{P}) = \max\{k : K_{k+1} \in \mathcal{P}\}$.

Given an arbitrary property \mathcal{P} , we define the *chromatic number* of \mathcal{P} as the minimum of the chromatic numbers of forbidden subgraphs of \mathcal{P} and we denote it by $\chi(\mathcal{P})$. It is clear, that for each additive hereditary property \mathcal{P} the value $\chi(\mathcal{P})$ is at least two.

The following results describe the structure of additive hereditary properties of graphs.

Theorem 1 [1]. *Let \mathbb{L} be the set of all hereditary properties. Then (\mathbb{L}, \subseteq) is a complete and distributive lattice in which the join and the meet correspond to the set-union and the set-intersection, respectively.*

Theorem 2 [1]. *For every nonnegative k , $\mathbb{L}_k = \{\mathcal{P} \in \mathbb{L} | c(\mathcal{P}) = k\}$ is a complete distributive sublattice of (\mathbb{L}, \subseteq) with the least element \mathcal{O}_k and the greatest element \mathcal{I}_k .*

2. TWO EXTREMAL GRAPH PROBLEMS

Many problems in graph theory involve optimization. One of them could be formulated in the following way: for a graph of given order a certain type of subgraphs is prohibited, and one is to determine the maximum possible number of edges in such a graph. A problem of this type was first formulated by Turán and his original problem asked for the maximum number of edges

in any graph of order n which does not contain the complete graph K_p (i.e., he was interested in the number $\text{ex}(n, \mathcal{I}_{p-2})$, see [2], [3], [9], [10], [12], [13]).

A general extremal problem, in our terminology, can be formulated as follows. Given a family $\mathbf{F}(\mathcal{P})$ of forbidden subgraphs, find the number

$$\text{ex}(n, \mathcal{P}) = \max\{|E(G)| : G \in \mathbf{M}(n, \mathcal{P})\}.$$

The set of \mathcal{P} -maximal graphs of order n with exactly $\text{ex}(n, \mathcal{P})$ edges is denoted by $\text{Ex}(n, \mathcal{P})$. The members of $\text{Ex}(n, \mathcal{P})$ are called \mathcal{P} -*extremal* graphs.

It is natural to investigate also the "opposite side", and therefore we define the number

$$\text{sat}(n, \mathcal{P}) = \min\{|E(G)| : G \in \mathbf{M}(n, \mathcal{P})\}.$$

By the symbol $\text{Sat}(n, \mathcal{P})$ we shall denote the set of all \mathcal{P} -maximal graphs on n vertices with $\text{sat}(n, \mathcal{P})$ edges. These graphs are called \mathcal{P} -*saturated*.

From the definitions immediately follows

Proposition 1. *Let $\mathcal{P}, \mathcal{P}_1$ and \mathcal{P}_2 be arbitrary hereditary properties and $G \in \mathbf{M}(n, \mathcal{P})$. Then*

1. $\text{sat}(n, \mathcal{P}) \leq |E(G)| \leq \text{ex}(n, \mathcal{P})$;
2. if $1 \leq n \leq c(\mathcal{P}) + 1$, then $\text{sat}(n, \mathcal{P}) = \text{ex}(n, \mathcal{P}) = \binom{n}{2}$;
3. $\text{ex}(n, \mathcal{P}) \leq \text{ex}(n+1, \mathcal{P})$ for every n ;
4. if $\mathcal{P}_1 \subseteq \mathcal{P}_2$, then $\text{ex}(n, \mathcal{P}_1) \leq \text{ex}(n, \mathcal{P}_2)$ for every n ;
5. $\text{ex}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = \max\{\text{ex}(n, \mathcal{P}_1), \text{ex}(n, \mathcal{P}_2)\}$ for $n \geq 1$;
6. $\text{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2) \leq \min\{\text{ex}(n, \mathcal{P}_1), \text{ex}(n, \mathcal{P}_2)\}$ for $n \geq 1$.

In [14] examples are presented, which demonstrate that unlike the number $\text{ex}(n, \mathcal{P})$, the behaviour of $\text{sat}(n, \mathcal{P})$ is not monotone in general.

The following theorems present some fundamental results of extremal graph theory. The symbol $\alpha(G)$ denotes the number of vertices in a maximum independent set of G .

Theorem 3 [11]. *If \mathcal{P} is a hereditary property with chromatic number $\chi(\mathcal{P})$, then*

$$\text{ex}(n, \mathcal{P}) = \left(1 - \frac{1}{\chi(\mathcal{P}) - 1}\right) \binom{n}{2} + o(n^2).$$

Theorem 4 [14]. *If \mathcal{P} is a given hereditary property and*

$$\begin{aligned} u = u(\mathcal{P}) &= \min\{|V(F)| - \alpha(F) - 1 : F \in \mathcal{P}\} \\ d = d(\mathcal{P}) &= \min\{|E(F')| : F' \subseteq F \in \mathbf{F}(\mathcal{P}) \text{ is induced by a set } S \cup \{x\}, \\ &\quad S \subseteq V(F) \text{ is independent and } |S| = |V(F)| - u - 1, \\ &\quad x \in V(F) \setminus S\}, \end{aligned}$$

then

$$\text{sat}(n, \mathcal{P}) \leq un + \frac{1}{2}(d-1)(n-u) - \binom{u+1}{2},$$

if n is large enough.

One can observe that in the case when the structure of $\mathbf{F}(\mathcal{P})$ is not known, the evaluation of the bound for $\text{sat}(n, \mathcal{P})$ is much more complicated as the evaluation of the bound for $\text{ex}(n, \mathcal{P})$. As a matter of fact, we can present hom-properties of graphs which were studied from this point of view in [4]. For that reason, in Section 3 we shall try to obtain another type of bounds for $\text{sat}(n, \mathcal{P})$.

However, as a consequence of the previous two theorems, we immediately have

Corollary 1. *If \mathcal{P} is a hereditary property of graphs and $\text{sat}(n, \mathcal{P}) = \text{ex}(n, \mathcal{P})$ for every positive n , then $\chi(\mathcal{P}) = 2$.*

3. INTERVALS OF MONOTONICITY

In spite of the fact that $\text{sat}(n, \mathcal{P})$ is not monotone, we can prove some inequalities and estimations using the properties of the lattice of all hereditary properties. It will be shown that the class of k -degenerate graphs plays a very important role since, $\text{sat}(n, \mathcal{D}_k) = \text{ex}(n, \mathcal{D}_k) = kn - \binom{k+1}{2}$ (see e.g. [5]).

Lemma 3. *Let $\mathcal{P}_1, \mathcal{P}_2$ be any hereditary properties and let $\mathcal{P}_1 \subseteq \mathcal{P}_2$. If $\text{sat}(n, \mathcal{P}_2) = \text{ex}(n, \mathcal{P}_2)$, then $\text{sat}(n, \mathcal{P}_1) \leq \text{sat}(n, \mathcal{P}_2)$.*

Proof. If $G \in \mathbf{M}(n, \mathcal{P}_1)$ then, by Lemma 1, there exists a graph $H \in \mathbf{M}(n, \mathcal{P}_2)$ such that $G \subseteq H$. Hence, $|E(G)| \leq |E(H)|$. Since $\text{ex}(n, \mathcal{P}_2) = |E(H)| = \text{sat}(n, \mathcal{P}_2)$, we obtain $|E(G)| \leq \text{sat}(n, \mathcal{P}_2)$. Therefore, $\text{sat}(n, \mathcal{P}_1) \leq |E(G)| \leq \text{sat}(n, \mathcal{P}_2)$. ■

Theorem 5. *If $\mathcal{O}_k \subseteq \mathcal{P} \subseteq \mathcal{D}_k$, $n \geq k + 1$, then $\text{sat}(n, \mathcal{P}) \leq kn - \binom{k+1}{2}$.*

Proof. As already pointed out, $\text{sat}(n, \mathcal{D}_k) = \text{ex}(n, \mathcal{D}_k) = kn - \binom{k+1}{2}$. Hence, by Lemma 3, we have $\text{sat}(n, \mathcal{P}) \leq kn - \binom{k+1}{2}$. ■

The following lemmas describe two other criteria of monotonicity in \mathbb{L} .

Lemma 4. *Let $\mathcal{P}_1, \mathcal{P}_2$ be any hereditary properties. Then*

$$\text{sat}(n, \mathcal{P}_1 \cup \mathcal{P}_2) \geq \min\{\text{sat}(n, \mathcal{P}_1), \text{sat}(n, \mathcal{P}_2)\}.$$

Proof. It is not difficult to see that $\mathbf{M}(n, \mathcal{P}_1 \cup \mathcal{P}_2)$ is a subset of $\mathbf{M}(n, \mathcal{P}_1) \cup \mathbf{M}(n, \mathcal{P}_2)$. Thus, $\text{sat}(n, \mathcal{P}_1 \cup \mathcal{P}_2)$ cannot be less than the minimum of $\text{sat}(n, \mathcal{P}_2)$ and $\text{sat}(n, \mathcal{P}_1)$. ■

Lemma 5. *Let \mathcal{P}_1 and \mathcal{P}_2 be any hereditary properties of graphs, $\mathcal{P}_1 \subseteq \mathcal{P}_2$, and let G be a graph of order n . If $G \in \mathcal{P}_1$ and G is \mathcal{P}_2 -saturated, then $\text{sat}(n, \mathcal{P}_1) \leq \text{sat}(n, \mathcal{P}_2)$.*

Proof. Lemma 2 yields $G \in \mathbf{M}(n, \mathcal{P}_1)$. Hence, by an application of Statement (1) of Proposition 1, we get $\text{sat}(n, \mathcal{P}_1) \leq |E(G)| = \text{sat}(n, \mathcal{P}_2)$. ■

Theorem 5 provides an upper bound for $\text{sat}(n, \mathcal{P})$ for the first part of interval $(\mathcal{O}_k, \mathcal{I}_k)$ in \mathbb{L}_k . The next theorem covers the rest of this interval. In order to prove it, we have to recall that in [6] it was proved that for any $F \in \mathbf{F}(\mathcal{D}_k)$ holds $\delta(F) \geq k + 1$.

Theorem 6. *If $\mathcal{D}_k \subseteq \mathcal{P} \subseteq \mathcal{I}_k$, $n \geq k + 1$, then $\text{sat}(n, \mathcal{P}) \leq kn - \binom{k+1}{2}$.*

Proof. Since $c(\mathcal{P}) = k$, we observe that $K_{k+2} \notin \mathcal{P}$. Hence, by Lemma 1, there exist graphs $F \in \mathbf{F}(\mathcal{D}_k)$ and $H \in \mathbf{F}(\mathcal{P})$ such that $F \subseteq H \subseteq K_{k+2}$. But, as it was mentioned above, $\delta(F) \geq k + 1$ and therefore $F = H = K_{k+2}$. In addition, according to the definition of $\mathbf{F}(\mathcal{P})$, no graph of $\mathbf{F}(\mathcal{P})$ is properly contained in K_{k+2} , which implies $|V(F)| \geq k + 2$ for any $F \in \mathbf{F}(\mathcal{P})$.

Now, let us define the graph G_n^k with the vertex set $V(G_n^k) = \{v_1, v_2, \dots, v_n\}$ in the following way (the symbol $N(u)$ stands for the neighbourhood of the vertex u):

$$\begin{aligned} N(v_i) &= \{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}, & i &= 1, 2, \dots, k, \\ N(v_i) &= \{v_1, v_2, \dots, v_k\}, & i &= k + 1, k + 2, \dots, n. \end{aligned}$$

The graph G_n^k does not contain a subgraph isomorphic to K_{k+2} , but it is easy to see that after adding any edge $e \in E(\overline{G_n^k})$ a copy of K_{k+2} must

appear in $G_n^k + e$. Hence, $G_n^k \in \mathbf{M}(n, \mathcal{I}_k)$. Furthermore, $G_n^k \in \mathcal{D}_k$ and then, applying Lemma 2, $G_n^k \in \mathbf{M}(n, \mathcal{P})$. This implies, using Lemma 5, Proposition 1, that $\text{sat}(n, \mathcal{P}) \leq |E(G_n^k)| = kn - \binom{k+1}{2}$. ■

4. SOME ESTIMATIONS OF $\text{sat}(n, \mathcal{P})$ AND $\text{ex}(n, \mathcal{P})$

In the previous section we have established a bound for $\text{sat}(n, \mathcal{P})$ in the part of the interval $(\mathcal{O}_k, \mathcal{I}_k)$ of the sublattice \mathbb{L}_k . The following theorem presents the exact value of $\text{sat}(n, \mathcal{P})$ in one specific case. By the invariant $\kappa(\mathcal{P})$ we understand the minimum of the numbers $\kappa(F)$, the vertex-connectivity number of F , running over all graphs F from $\mathbf{F}(\mathcal{P})$. We shall use the fact, proved in [7], that for any \mathcal{P} -maximal graph G the value $\kappa(G)$ is at least $\kappa(\mathcal{P}) - 1$.

Theorem 7. *Let \mathcal{P} be a hereditary property and let $\mathcal{D}_1 \subseteq \mathcal{P} \subseteq \mathcal{I}_1$. If $\kappa(\mathcal{P}) \geq 1$, then $\text{sat}(n, \mathcal{P}) = n - 1$.*

Proof. By Theorem 6, we have $\text{sat}(n, \mathcal{P}) \leq n - 1$. An application of the fact, that the minimum degree of a graph from $\mathbf{F}(\mathcal{D}_1)$ is 2, and Lemma 1 yields that any $F \in \mathbf{F}(\mathcal{P})$ has a subgraph isomorphic to C_n for some $n \geq 3$ (the symbol C_n stands for the cycle on n vertices). We distinguish two cases.

Case 1. Let $\kappa(\mathcal{P}) = 1$. Suppose indirectly that $\text{sat}(n, \mathcal{P}) \leq n - 2$ for some n . Then there exists a graph $G \in \mathbf{M}(n, \mathcal{P})$ with at most $n - 2$ edges. It is easy to see that G is disconnected. Let us denote by G_1, G_2, \dots, G_s , $s \geq 2$, the components of G and let $r_i = |V(G_i)|$ for $i = 1, 2, \dots, s$. Since each G_i has at least $r_i - 1$ edges and $\sum_{i=1}^s r_i = n$, it follows that at least two components of G , say G_1, G_2 , are trees. Then after adding any edge $e = \{u, v\}$, $u \in V(G_1)$, $v \in V(G_2)$, some $F \in \mathbf{F}(\mathcal{P})$ must appear in $G + e$. Since $\kappa(G) = 1$, we obtain $F \subseteq (G_1 \cup G_2) + e$. But $(G_1 \cup G_2) + e$ is a tree which contradicts the fact that F contains a cycle.

Case 2. Let $\kappa(\mathcal{P}) \geq 2$. If $G \in \mathbf{M}(n, \mathcal{P})$ then G is connected. Hence G has at least $n - 1$ edges. Therefore $\text{sat}(n, \mathcal{P}) = n - 1$. ■

The set of k -degenerate graphs is one with $\text{sat}(n, \mathcal{P}) = \text{ex}(n, \mathcal{P})$. It is widely known that the properties \mathcal{T}_2 (to be an outerplanar graph) and \mathcal{T}_3 (to be a planar graph) are other examples of such properties. We show that such properties have an exceptional position in the lattice \mathbb{L} of all hereditary properties.

Lemma 6. *Let $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$ be any hereditary properties of graphs and let $f : \{1, 2, \dots\} \rightarrow \{0, 1, \dots\}$ be a mapping. If $\text{ex}(n, \mathcal{P}_1) = \text{ex}(n, \mathcal{P}_3) = f(n)$, then $\text{sat}(n, \mathcal{P}_2) \leq f(n)$ and $\text{ex}(n, \mathcal{P}_2) = f(n)$.*

Proof. By Statement (4) of Proposition 1, we have $f(n) = \text{ex}(n, \mathcal{P}_1) \leq \text{ex}(n, \mathcal{P}_2) \leq \text{ex}(n, \mathcal{P}_3) = f(n)$, which implies that $\text{ex}(n, \mathcal{P}_2) = f(n)$. Since $\text{sat}(n, \mathcal{P}_2) \leq \text{ex}(n, \mathcal{P}_2)$ the assertion $\text{sat}(n, \mathcal{P}_2) \leq f(n)$ is also valid. ■

Theorem 8. *If \mathcal{P} is a hereditary property, $\mathcal{T}_2 \subseteq \mathcal{P} \subseteq \mathcal{D}_2$, then $\text{sat}(n, \mathcal{P}) \leq 2n - 3$ and $\text{ex}(n, \mathcal{P}) = 2n - 3$ for $n \geq 3$.*

Proof. The proof follows from the fact that $\mathcal{T}_2 \subseteq \mathcal{D}_2$ and the number of edges of all \mathcal{T}_2 -maximal and \mathcal{D}_2 -maximal graphs of order $n \geq 3$ is exactly $2n - 3$. ■

Lemma 7. *Let \mathcal{P}_1 and \mathcal{P}_2 be any hereditary properties of graphs and let $f : \{1, 2, \dots\} \rightarrow \{0, 1, \dots\}$ be a mapping. If $\text{sat}(n, \mathcal{P}_1) = \text{sat}(n, \mathcal{P}_2) = \text{ex}(n, \mathcal{P}_1) = \text{ex}(n, \mathcal{P}_2) = f(n)$, then*

1. $\text{sat}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = \text{ex}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = f(n)$;
2. $\text{sat}(n, \mathcal{P}_1 \cap \mathcal{P}_2) \leq f(n)$ and $\text{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2) \leq f(n)$.

Furthermore, if there exists a graph $G \in \mathbf{M}(n, \mathcal{P}_1) \cap \mathbf{M}(n, \mathcal{P}_2)$, then $\text{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2) = f(n)$.

Proof. (1) From the fact $\mathbf{M}(n, \mathcal{P}_1 \cup \mathcal{P}_2) \subseteq \mathbf{M}(n, \mathcal{P}_1) \cup \mathbf{M}(n, \mathcal{P}_2)$ it follows that $\text{sat}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = \text{ex}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = f(n)$.

(2) By Proposition 1, we have $\text{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2) \leq \min\{\text{ex}(n, \mathcal{P}_1), \text{ex}(n, \mathcal{P}_2)\} = f(n)$. Since $\text{sat}(n, \mathcal{P}_1 \cap \mathcal{P}_2) \leq \text{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2)$, we obtain the desired inequality.

Moreover, if there exists a graph $G \in \mathbf{M}(n, \mathcal{P}_1) \cap \mathbf{M}(n, \mathcal{P}_2)$, then $G \in \mathbf{M}(n, \mathcal{P}_1 \cap \mathcal{P}_2)$. Clearly, $|E(G)| = f(n)$. It immediately follows that $\text{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2) = f(n)$. ■

It is easy to see that \mathcal{T}_3 and \mathcal{D}_3 are incomparable in the lattice \mathbf{L} . So we can examine the lattice interval between $\mathcal{T}_3 \cap \mathcal{D}_3$ and $\mathcal{T}_3 \cup \mathcal{D}_3$.

Lemma 8. *If n is a positive integer, $n \geq 4$, then*

1. $\text{sat}(n, \mathcal{T}_3 \cup \mathcal{D}_3) = \text{ex}(n, \mathcal{T}_3 \cup \mathcal{D}_3) = 3n - 6$;
2. $\text{sat}(n, \mathcal{T}_3 \cap \mathcal{D}_3) \leq 3n - 6$ and $\text{ex}(n, \mathcal{T}_3 \cap \mathcal{D}_3) = 3n - 6$.

Proof. As $\text{ex}(n, \mathcal{T}_3) = \text{ex}(n, \mathcal{D}_3) = 3n - 6$ for $n \geq 4$, we have, by Lemma 7, that $\text{sat}(n, \mathcal{T}_3 \cup \mathcal{D}_3) = \text{ex}(n, \mathcal{T}_3 \cup \mathcal{D}_3) = 3n - 6$ and $\text{sat}(n, \mathcal{T}_3 \cap \mathcal{D}_3) \leq 3n - 6$.

It is easy to see that there exists a graph G with $3n - 6$ edges which is planar and 3-degenerate. It means $G \in \mathbf{M}(n, \mathcal{D}_3)$ and simultaneously

$G \in \mathbf{M}(n, \mathcal{T}_3)$. Hence, $G \in \mathbf{M}(n, \mathcal{D}_3 \cup \mathcal{T}_3)$ and $G \in \mathbf{M}(n, \mathcal{D}_3 \cap \mathcal{T}_3)$. Therefore, by Lemma 7, $\text{ex}(n, \mathcal{T}_3 \cup \mathcal{D}_3) = 3n - 6$. ■

The next theorem is an immediate consequence of the previous two lemmas.

Theorem 9. *Let \mathcal{P} be a hereditary property such that $\mathcal{T}_3 \cap \mathcal{D}_3 \subseteq \mathcal{P} \subseteq \mathcal{T}_3 \cup \mathcal{D}_3$. Then $\text{ex}(n, \mathcal{P}) = 3n - 6$ and $\text{sat}(n, \mathcal{P}) \leq 3n - 6$ for $n \geq 4$.*

5. REDUCIBLE HEREDITARY PROPERTIES

A generalization of a colouring of graphs leads us to the concept of reducible hereditary properties.

Given hereditary properties $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$, a *vertex* $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -*partition* of a graph $G \in \mathcal{I}$ is a partition (V_1, V_2, \dots, V_n) of $V(G)$ such that for each $i = 1, 2, \dots, n$ the induced subgraph $G[V_i]$ has the property \mathcal{P}_i . A property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n$ is defined as the set of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partition (for more details see [1], [8]).

The structure of extremal graphs with respect to reducible hereditary property is described by the following lemma.

Lemma 9. *If a graph G belongs to $\text{Ex}(n, \mathcal{P}_1 \circ \mathcal{P}_2)$, then for each $(\mathcal{P}_1, \mathcal{P}_2)$ -partition of $V(G)$ into two disjoint sets V_1, V_2 the following holds: the induced subgraph $G[V_1]$ is \mathcal{P}_1 -extremal, $G[V_2]$ is \mathcal{P}_2 -extremal and $G = G[V_1] + G[V_2]$.*

Proof. If G is $\mathcal{P}_1 \circ \mathcal{P}_2$ -extremal, then obviously for any $(\mathcal{P}_1, \mathcal{P}_2)$ -partition of $V(G)$ into V_1 and V_2 holds $G = G[V_1] + G[V_2]$ (otherwise we can add at least one edge, which is a contradiction to the extremality of G). Furthermore, if the graph $G[V_1]$ is not \mathcal{P}_1 -extremal, then there exists a graph $G^* \in \mathcal{P}_1$ of the same order with greater number of edges as $G[V_1]$. Clearly, $G^* + G[V_2] \in \mathcal{P}_1 \circ \mathcal{P}_2$ and moreover, $|E(G^* + G[V_2])| > |E(G[V_1] + G[V_2])|$, which is again a contradiction. Thereby $G[V_1]$ is \mathcal{P}_1 -extremal. Analogous arguments work for $G[V_2]$ and that is why $G[V_2]$ is a \mathcal{P}_2 -extremal graph. ■

As in [7] it was shown that $\chi(\mathcal{P}_1 \circ \mathcal{P}_2) = \chi(\mathcal{P}_1) + \chi(\mathcal{P}_2) - 1$, we immediately have

Theorem 10. *If $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ is a reducible hereditary property, then*

$$\text{ex}(n, \mathcal{R}) = \left(1 - \frac{1}{\chi(\mathcal{P}_1) + \chi(\mathcal{P}_2) - 2}\right) \binom{n}{2} + o(n^2).$$

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