

## UNIQUELY PARTITIONABLE GRAPHS

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### Abstract

Let  $\mathcal{P}_1, \dots, \mathcal{P}_n$  be properties of graphs. A  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partition of a graph  $G$  is a partition of the vertex set  $V(G)$  into subsets  $V_1, \dots, V_n$  such that the subgraph  $G[V_i]$  induced by  $V_i$  has property  $\mathcal{P}_i$ ;  $i = 1, \dots, n$ . A graph  $G$  is said to be uniquely  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable if  $G$  has exactly one  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partition. A property  $\mathcal{P}$  is called hereditary if every subgraph of every graph with property  $\mathcal{P}$  also has property  $\mathcal{P}$ . If every graph that is a disjoint union of two graphs that have property  $\mathcal{P}$  also has property  $\mathcal{P}$ , then we say that  $\mathcal{P}$  is additive. A property  $\mathcal{P}$  is called degenerate if there exists a bipartite graph that does not have property  $\mathcal{P}$ . In this paper, we prove that if  $\mathcal{P}_1, \dots, \mathcal{P}_n$  are degenerate, additive, hereditary properties of graphs, then there exists a uniquely  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable graph.

**Keywords:** hereditary property of graphs, additivity, reducibility, vertex partition.

**1991 Mathematics Subject Classification:** 05C15, 05C70.

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<sup>1</sup>Research supported by the South African Foundation for Research Development.

<sup>2</sup>Research supported in part by the Slovak VEGA grant.

## 1. NOTATION AND BACKGROUND

All graphs considered in this paper are finite and simple. In general, we follow the notation and terminology of [15].

We denote the set of all mutually nonisomorphic graphs by  $\mathcal{I}$ . Each nonempty subset  $\mathcal{P} \subseteq \mathcal{I}$  is also said to be a *property of graphs*. A property  $\mathcal{P}$  is said to be *hereditary* if  $G \in \mathcal{P}$  and  $H \subseteq G$  implies  $H \in \mathcal{P}$ . A property  $\mathcal{P}$  is *additive* if  $G_1, G_2 \in \mathcal{P}$  implies that the disjoint union  $G_1 \cup G_2$  is also in  $\mathcal{P}$ . We shall denote the set of all hereditary properties by  $\mathbb{L}$ , and the set of all additive, hereditary properties by  $\mathbb{L}^a$ . We list some additive, hereditary properties in Table 1. (We use the notation of [6] for most of them).

Table 1

The property	The graphs which have the property
$\mathcal{O}$	$G \in \mathcal{I}$ ; $G$ is totally disconnected
$\mathcal{S}_k$	$G \in \mathcal{I}$ ; $\Delta(G) \leq k$
$\mathcal{W}_k$	$G \in \mathcal{I}$ ; the length of the longest path in $G$ does not exceed $k$
$\mathcal{D}_k$	$G \in \mathcal{I}$ ; $G$ is $k$ -degenerate i.e., $\delta(H) \geq k$ for $H \subseteq G$
$\mathcal{T}_k$	$G \in \mathcal{I}$ ; $G$ contains no subgraph homeomorphic to $K_{k+2}$ or $K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}$
$\mathcal{I}_k$	$G \in \mathcal{I}$ ; $G$ does not contain $K_{k+2}$ as a subgraph

Any hereditary property  $\mathcal{P}$  is uniquely determined by the set

$$\mathbf{F}(\mathcal{P}) = \{G \in \mathcal{I} \mid G \notin \mathcal{P} \text{ but each proper subgraph of } G \text{ belongs to } \mathcal{P}\}$$

of *minimal forbidden subgraphs* (see [6], [14], [16], [18]), or by the set of so-called  $\mathcal{P}$ -*maximal* graphs

$$\mathbf{M}(\mathcal{P}) = \{G \in \mathcal{P} \mid G + e \notin \mathcal{P} \text{ for every } e \in \overline{G}\},$$

(see [6], [24], [29]).

The *join* of two vertex disjoint graphs  $G_1$  and  $G_2$  is obtained by joining every vertex of  $G_1$  to every vertex of  $G_2$ , and is denoted by  $G_1 + G_2$ .

A graph  $G$  is said to be  $\mathcal{P}$ -*strict* if  $G \in \mathcal{P}$  and  $G + K_1 \notin \mathcal{P}$ .

Let  $\mathcal{P}$  be a hereditary property,  $\mathcal{P} \neq \mathcal{I}$ . Then there is a nonnegative integer  $c(\mathcal{P})$  such that  $K_{c(\mathcal{P})+1} \in \mathcal{P}$  but  $K_{c(\mathcal{P})+2} \notin \mathcal{P}$ , called the *completeness* of  $\mathcal{P}$ . Clearly, every  $\mathcal{P}$ -maximal graph  $G$  with  $|V(G)| \geq c(\mathcal{P}) + 1$  is  $\mathcal{P}$ -strict.

For any property  $\mathcal{P}$  we define the minimum degree of  $\mathcal{P}$  as

$$\delta(\mathcal{P}) = \min\{\delta(G) \mid G \in \mathbf{F}(\mathcal{P})\},$$

and the chromatic number of  $\mathcal{P}$  as

$$\chi(\mathcal{P}) = \min\{\chi(G) \mid G \in \mathbf{F}(\mathcal{P})\}.$$

Let  $\mathcal{P}_1, \dots, \mathcal{P}_n$  be properties of graphs. A  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partition of a graph  $G$  is a partition  $\{V_1, \dots, V_n\}$  of  $V(G)$  such that the subgraph  $G[V_i]$  induced by  $V_i$  has property  $\mathcal{P}_i$  for  $i = 1, \dots, n$ . The property  $\mathcal{R} = \mathcal{P}_1 \circ \dots \circ \mathcal{P}_n$  is defined as the set of all graphs that have a  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partition.

If  $\mathcal{P}_1 = \dots = \mathcal{P}_n$ , the property  $\mathcal{P}_1 \circ \dots \circ \mathcal{P}_n$  will be denoted by  $\mathcal{P}^n$ . For example, the class of all  $n$ -colourable graphs is denoted by  $\mathcal{O}^n$ .

If there exist properties  $\mathcal{P}$  and  $\mathcal{Q}$  such that  $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$ , then  $\mathcal{R}$  is said to be a *reducible* property and  $\mathcal{P}$ , and  $\mathcal{Q}$  are said to *divide*  $\mathcal{R}$ ; otherwise  $\mathcal{R}$  is called *irreducible* (see e.g., [6], [20], [23]). Different generalizations of regular colouring of the vertices of graphs (see e.g. [1], [2], [8], [9], [10], [11], [12], [20], [21], [22], [25], [26], [31]) can be expressed using the notion of reducible properties.

We shall need the following two lemmas concerning reducible properties.

**Lemma 1.** *If  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are (additive) hereditary properties of graphs, then the property  $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$  is also (additive) hereditary.*

**Lemma 2.** *Let  $\mathcal{P}_1$  and  $\mathcal{P}_2$  be hereditary properties of graphs and let  $G$  be a  $\mathcal{P}_1 \circ \mathcal{P}_2$ -maximal graph. If  $\{V_1, V_2\}$  is any  $(\mathcal{P}_1, \mathcal{P}_2)$ -partition of  $G$ , then*

$$G = G[V_1] + G[V_2]$$

*and the graph  $G[V_i]$  are  $\mathcal{P}_i$ -maximal,  $i = 1, 2$ .*

**Proof.** Suppose that there exists an edge  $e = (x, y)$  such that  $x \in V_1$  and  $y \in V_2$  and  $e \notin E(G)$ . Then the graph  $G + e \in \mathcal{P}_1 \circ \mathcal{P}_2$ , contradicting our assumption that  $G \in \mathbf{M}(\mathcal{P}_1 \circ \mathcal{P}_2)$ . This proves that  $G = G[V_1] + G[V_2]$ .

Now suppose  $G[V_1]$  is not  $\mathcal{P}_1$ -maximal. Then  $G[V_1] + e \in \mathcal{P}_1$  for some  $e \in E(\overline{G[V_1]})$ . But then, again,  $G + e \in \mathcal{P}_1 \circ \mathcal{P}_2$ . This contradiction proves that  $G[V_1]$  is  $\mathcal{P}_1$ -maximal. Likewise,  $G[V_2]$  is  $\mathcal{P}_2$ -maximal. ■

A graph  $G \in \mathcal{P}_1 \circ \dots \circ \mathcal{P}_n$  is said to be *uniquely*  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -*partitionable* if  $G$  has exactly one  $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partition. The set of all uniquely  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable graphs will be denoted by  $\mathbf{U}(\mathcal{P}_1 \circ \dots \circ \mathcal{P}_n)$ , e.g.,  $\mathbf{U}(\mathcal{O}^n)$  denotes the set of all uniquely  $n$ -colourable graphs (see [4], [19], [17]);  $\mathbf{U}(\mathcal{S}_k^n)$  denotes the set of all uniquely  $(m, k)^\Delta$ -colourable graphs (see [12], [13], [32]);  $\mathbf{U}(\mathcal{W}_k^n)$  has been studied in [12], [3] and  $\mathbf{U}(\mathcal{D}_k^n)$  in [5], [27], and  $\mathbf{U}(\mathcal{I}_k^n)$  in [7], [12]. The basic properties of  $\mathbf{U}(\mathcal{P}^n)$  have been investigated in [5], [23]. Another generalization of uniquely colourable graphs was introduced by X. Zhu in [33].

The notion of *degenerate* hereditary property appeared with regards to the famous Erdős-Simonovits formula

$$\text{ext}(n, \mathcal{P}) = \frac{\chi(\mathcal{P}) - 2}{\chi(\mathcal{P}) - 1} \binom{n}{2} + o(n^2),$$

where

$$\text{ext}(n, \mathcal{P}) = \max\{|E(G)| \mid G \in \mathcal{P} \text{ and } |V(G)| = n\},$$

A property  $\mathcal{P} \in \mathbb{L}^a$  is said to be *degenerate* if  $\chi(\mathcal{P}) = 2$ , i.e., if  $\mathbf{F}(\mathcal{P})$  contains some bipartite graph (see [29], [30]). Obviously,  $\mathcal{O}, \mathcal{S}_k, \mathcal{Q}_k, \mathcal{O}_k, \mathcal{D}_k$  and  $\mathcal{T}_k$  are degenerate properties of graphs, but the property  $\mathcal{I}_k$  is not degenerate.

In [23] it is proved that if the property  $\mathcal{P}$  is a reducible property of graphs, then  $\mathbf{U}(\mathcal{P}^n) = \emptyset$  and we also proved that  $\mathbf{U}(\mathcal{P}^n) \neq \emptyset$  for every degenerate property  $\mathcal{P}$ , which means that every degenerate property is irreducible. In Section 4 of this paper, we generalize this result by proving that  $\mathbf{U}(\mathcal{P}_1 \circ \dots \circ \mathcal{P}_n) \neq \emptyset$  if  $\mathcal{P}_1, \dots, \mathcal{P}_n$  are degenerate, additive, hereditary properties.

In Section 2 we present some basic properties of uniquely  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable graphs, generalizing results known to hold for uniquely colourable graphs.

In Section 3 we provide a necessary and sufficient condition for one hereditary property to be divisible by another. This result is used to prove our main result, Theorem 3, which gives a necessary and sufficient condition for the existence of uniquely  $\mathcal{P} \circ \mathcal{Q}$ -partitionable graphs, when  $\mathcal{P}$  and  $\mathcal{Q}$  are additive, hereditary properties and  $\mathcal{Q}$  is degenerate.

## 2. BASIC PROPERTIES OF UNIQUELY PARTITIONABLE GRAPHS

The results on uniquely  $\mathcal{P}^n$ -partitionable graphs obtained in [23] can be directly generalized to obtain the properties of uniquely  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable graphs presented in the following two theorems.

**Theorem 1.** *Let  $\mathcal{P}_1, \dots, \mathcal{P}_n$  be hereditary properties of graphs,  $n \geq 2$ . If  $G$  is a uniquely  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable graph and  $\{V_1, \dots, V_n\}$  is the unique  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partition of  $V(G)$ , then*

1.  $G \notin \mathcal{P}_1 \circ \dots \circ \mathcal{P}_{j-1} \circ \mathcal{P}_{j+1} \circ \dots \circ \mathcal{P}_n$  for  $j = 1, \dots, n$ ,
2. the subgraphs  $G[V_i]$  are  $\mathcal{P}_i$ -strict,  $i = 1, 2, \dots, n$ ,
3. if  $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ , then  $V_{i_1} \cup \dots \cup V_{i_k}$  induces a uniquely  $(\mathcal{P}_{i_1}, \dots, \mathcal{P}_{i_n})$ -partitionable subgraph of  $G$ ,
4.  $\delta(G) \geq \max_j \sum_{i=1, i \neq j}^n \delta(\mathcal{P}_i)$ ,
5.  $|V(G)| \geq \sum_{i=1}^n (c(\mathcal{P}_i) + 2) - 1$ ,
6. the graph  $G = G[V_1] + \dots + G[V_n]$  is uniquely  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable.

**Theorem 2.** *Let  $\mathcal{P}_1, \dots, \mathcal{P}_n$  be hereditary properties of graphs. If  $G \in \mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n$  and  $U(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n) \neq \emptyset$ , then  $G$  is an induced subgraph of some uniquely  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable graph.*

### 3. DIVISIBILITY AND UNIQUELY $(\mathcal{P}, \mathcal{Q})$ -PARTITIONABLE GRAPHS

**Lemma 3.** *If  $\mathcal{P}$  and  $\mathcal{Q}$  are properties of graphs such that one of the following holds:*

1.  $\mathcal{P}$  divides  $\mathcal{Q}$
2.  $\mathcal{Q}$  divides  $\mathcal{P}$
3. there exists a property  $\mathcal{S}$  such that  $\mathcal{S}$  divides both  $\mathcal{P}$  and  $\mathcal{Q}$ ,

*then  $U(\mathcal{P} \circ \mathcal{Q}) = \emptyset$ .*

**Proof.** 1. Suppose  $\mathcal{Q} = \mathcal{P} \circ \mathcal{Q}^*$  for some property  $\mathcal{Q}^*$ . Let  $G \in \mathcal{P} \circ \mathcal{Q}$  and let  $\{V_1, V_2\}$  be a  $(\mathcal{P}, \mathcal{Q})$ -partition of  $G$ , with  $V_1, V_2 \neq \emptyset$ . Since  $G[V_2] \in \mathcal{Q} = \mathcal{P} \circ \mathcal{Q}^*$ , there exists a partition  $\{V_{21}, V_{22}\}$  of  $G[V_2]$ , with  $V_{21}, V_{22} \neq \emptyset$ , such that  $G[V_{21}] \in \mathcal{P}$  and  $G[V_{22}] \in \mathcal{Q}^*$ . But then  $G[V_1 \cup V_{22}] \in \mathcal{P} \circ \mathcal{Q}^* = \mathcal{Q}$ , and thus  $\{V_{21}, V_1 \cup V_{22}\}$  is a  $(\mathcal{P}, \mathcal{Q})$ -partition of  $G$  different from  $\{V_1, V_2\}$ , which implies that  $G$  is not uniquely  $(\mathcal{P}, \mathcal{Q})$ -partitionable.

Cases (2) and (3) can be proved in an analogous way. ■

If  $\mathcal{P}$  and  $\mathcal{Q}$  are additive hereditary properties and  $\mathcal{Q}$  is also degenerate, then converse of Lemma 1 also holds. In order to prove this, we introduce the concept of an extendible set.

Let  $\mathcal{P}$  and  $\mathcal{Q}$  be hereditary properties of graphs and let  $G \in \mathcal{P}$ . If  $S$  is a subset of  $V(G)$  such that  $G[S] \in \mathcal{Q}$  and for every graph  $T \in \mathcal{Q}$  the graph  $T + (G - S) \in \mathcal{P}$ , then  $S$  is said to be a  $(\mathcal{Q}, \mathcal{P})$ -extendible set of  $G$ .

We shall need the following lemma.

**Lemma 4.** *Let  $\mathcal{P}$  and  $\mathcal{Q}$  be hereditary properties of graphs. If  $H$  is a graph with property  $\mathcal{P}$  that has no  $(\mathcal{Q}, \mathcal{P})$ -extendible set, then there exists a  $\mathcal{P}$ -strict graph  $G$  such that  $G$  has no  $(\mathcal{Q}, \mathcal{P})$ -extendible set.*

**Proof.** Let  $H$  be a graph with property  $\mathcal{P}$  such that  $H$  has no  $(\mathcal{Q}, \mathcal{P})$ -extendible set. Let  $G$  be a  $\mathcal{P}$ -strict graph such that  $H \subseteq G$ . Suppose, to the contrary, that  $G$  contains a  $(\mathcal{Q}, \mathcal{P})$ -extendible set  $S$ . Let  $S' = S \cap V(H)$ . Let  $T$  be any graph with property  $\mathcal{Q}$ . Then

$$T + (G - S) \in \mathcal{P}.$$

Since  $T + (H - S') \subseteq T + (H - S)$  and  $\mathcal{P}$  is hereditary, this implies that  $T + (H - S') \in \mathcal{P}$ , so that  $S'$  is an extendible set of  $H$ . ■

We have the following connection between divisibility and the existence of an extendible set.

**Theorem 3.** *Let  $\mathcal{P}$  and  $\mathcal{Q}$  be hereditary properties of graphs. Then  $\mathcal{Q}$  divides  $\mathcal{P}$  if and only if every  $\mathcal{P}$ -maximal graph contains a  $(\mathcal{Q}, \mathcal{P})$ -extendible set.*

**Proof.** Suppose  $\mathcal{Q}$  divides  $\mathcal{P}$ . Then there is a property  $\mathcal{P}^*$  such that  $\mathcal{P} = \mathcal{Q} \circ \mathcal{P}^*$ . Let  $G \in \mathcal{P}$  and let  $\{V_1, V_2\}$  be a  $(\mathcal{Q}, \mathcal{P}^*)$ -partition of  $G$ . Let  $T$  be any graph with property  $\mathcal{Q}$ . Then  $\{V(T), V_2\}$  is a  $(\mathcal{Q}, \mathcal{P}^*)$ -partition of  $T + G[V_2]$ , and hence  $T + G[V_2] \in \mathcal{P}$ . Since  $G[V_2] = G - V_1$ , this proves that  $V_1$  is a  $(\mathcal{Q}, \mathcal{P})$ -extendible set of  $G$ .

To prove the converse, suppose every  $\mathcal{P}$ -maximal graph contains a  $(\mathcal{Q}, \mathcal{P})$ -extendible set. Let

$$\mathcal{S}(G) = \{S \subseteq V(G) \mid S \text{ is an extendible set of } G\}$$

and put

$$\mathcal{P}' = \{G - S \mid G \in \mathbf{M}(\mathcal{P}), S \in \mathcal{S}(G)\}.$$

Now let  $\mathcal{P}^*$  be the property consisting of all subgraphs of graphs in  $\mathcal{P}'$ . Then  $\mathcal{P}^*$  is a hereditary property. We shall prove that  $\mathcal{P} = \mathcal{Q} \circ \mathcal{P}^*$ .

Suppose  $G \in \mathbf{M}(\mathcal{P})$ . Then, by our assumption,  $G$  has a  $(\mathcal{Q}, \mathcal{P})$ -extendible set. Let  $S$  be such a set. Then  $G - S \in \mathcal{P}'$ , by the definition

of  $\mathcal{P}^*$ . Thus  $\{S, G - S\}$  is a  $(\mathcal{Q}, \mathcal{P}^*)$ -partition of  $G$ , so that  $G \in \mathcal{Q} \circ \mathcal{P}^*$ . This proves that  $\mathbf{M}(\mathcal{P}) \subseteq \mathcal{Q} \circ \mathcal{P}^*$ . But  $\mathcal{Q} \circ \mathcal{P}^*$  is a hereditary property by Lemma 2, and hence  $\mathcal{P} \subseteq \mathcal{Q} \circ \mathcal{P}^*$ .

Now suppose  $G \in \mathbf{M}(\mathcal{Q} \circ \mathcal{P}^*)$ . Let  $\{V_1, V_2\}$  be a  $(\mathcal{Q}, \mathcal{P}^*)$ -partition of  $G$ . Then it follows from Lemma 2 that  $G[V_1] \in \mathbf{M}(\mathcal{Q})$ ,  $G[V_2] \in \mathbf{M}(\mathcal{P}^*)$  and  $G = G[V_1] + G[V_2]$ . By the definition of  $\mathcal{P}^*$  there exists a  $\mathcal{P}$ -maximal graph  $F$  and a  $(\mathcal{Q}, \mathcal{P})$ -extendible set  $S$  of  $F$  such that  $G[V_2] \subseteq F - S$ . But then, since  $G[V_1] \in \mathcal{Q}$ , we have  $G[V_1] + F - S \in \mathcal{P}$ . But  $G \subseteq G[V_1] + F - S$ , and hence  $G \in \mathcal{P}$ . This proves that  $\mathcal{Q} \circ \mathcal{P}^* \subseteq \mathcal{P}$ . ■

**Theorem 4.** *Let  $\mathcal{P}, \mathcal{Q} \in \mathbb{L}^a$  and let  $\mathcal{Q}$  be a degenerate property. Then  $U(\mathcal{Q} \circ \mathcal{P}) \neq \emptyset$  if and only if  $\mathcal{Q}$  does not divide  $\mathcal{P}$ .*

**Proof.** If  $U(\mathcal{Q} \circ \mathcal{P}) \neq \emptyset$  then, by Lemma 3,  $\mathcal{Q}$  does not divide  $\mathcal{P}$ .

To prove the converse, suppose  $\mathcal{Q}$  does not divide  $\mathcal{P}$ . Then it follows from Theorem 3 and Lemma 4 that there exists a  $\mathcal{P}$ -strict graph  $H$  that contains no  $(\mathcal{Q}, \mathcal{P})$ -extendible set. Let

$$Z = \{S \mid S \subseteq V(H) \text{ and } H[S] \in \mathcal{Q}\}.$$

Then, for every  $S \in Z$ , there exists a  $\mathcal{Q}$ -strict graph  $T(S)$  such that

$$T(S) + (H - S) \notin \mathcal{P}.$$

Now let

$$T = \cup_{S \in Z} T(S).$$

Since  $\mathcal{Q}$  is a degenerate property, there is an integer  $q$  such that  $K_{q,q} \notin \mathcal{Q}$ . Let

$$G_1 = qT, \quad G_2 = qH, \quad \text{and} \quad G = G_1 + G_2.$$

Since  $\mathcal{P}$  and  $\mathcal{Q}$  are additive properties,  $G_1 \in \mathcal{P}$  and  $G_2 \in \mathcal{Q}$ , and thus  $G \in \mathcal{P} \circ \mathcal{Q}$ .

Now let  $\{W_1, W_2\}$  be any  $(\mathcal{P}, \mathcal{Q})$ -partition of  $G$ . Suppose each of the  $q$  copies of  $H$  in  $G_2$  has at least one vertex in  $W_1$ . Then

$$|V(G_2) \cap W_1| \geq q.$$

Now let  $H_0$  be a specific copy of  $H$  in  $G_2$ , and let  $S_0 = V(H_0) \cap W_1$ . Then  $H_0[S_0] \in \mathcal{Q}$  and hence, by the definition of  $T$ , we have

$$T + H_0 - S_0 \notin \mathcal{P}.$$

Since  $V(H_0) - S_0 \in W_2$ , it follows that none of the  $q$  copies of  $T$  in  $G_1$  has all its vertices in  $W_2$ . Thus

$$|V(G_1) \cap W_1| \geq q.$$

But then  $K_{q,q} \subseteq G[W_1]$ . This contradiction proves that at least one of the  $q$  copies of  $H$  in  $G_2$  has all its vertices in  $W_2$ . Since  $H$  is  $\mathcal{P}$ -strict, it follows that  $W_2 \cap V(G_1) = \emptyset$ . But  $G_1$  is  $\mathcal{Q}$ -strict, and hence  $W_1 = G(V_1)$ , which implies that  $\{V(G_1), V(G_2)\}$  is the only  $(\mathcal{Q}, \mathcal{P})$ -partition of  $G$ . Thus  $G \in \mathcal{U}(\mathcal{Q} \circ \mathcal{P})$ .  $\blacksquare$

#### 4. CONSTRUCTION OF UNIQUELY $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -PARTITIONABLE GRAPHS FOR DEGENERATE PROPERTIES

Uniquely  $(\mathcal{P}^n)$ -partitionable graphs have been proved to exist for several specific degenerate properties  $\mathcal{P}$  (see [5], [23], [28]). The following theorem generalizes those results.

**Theorem 5.** *Let  $\mathcal{P}_1, \dots, \mathcal{P}_n$ , be degenerate, additive, hereditary properties of graphs. Then there exists a uniquely  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable graph.*

**Proof.** We may assume, without loss of generality, that the properties  $\mathcal{P}_1, \dots, \mathcal{P}_n$  are ordered in such a way that  $\mathcal{P}_i \not\subseteq \mathcal{P}_j$  if  $i < j$  and, if  $\mathcal{P}_i = \mathcal{P}_j$  and  $i < k < j$ , then  $\mathcal{P}_i = \mathcal{P}_k$ . Then there exist graphs  $H_1, \dots, H_n$  such that  $H_i$  is  $\mathcal{P}_i$ -strict for  $i = 1, \dots, n$  and, if  $i < j$ , then  $H_i \not\subseteq \mathcal{P}_j$  unless  $\mathcal{P}_i = \mathcal{P}_j$ . Since  $\mathcal{P}_1, \dots, \mathcal{P}_n$  are degenerate properties, there exists an integer  $q$  such that  $K_{q,q} \not\subseteq \mathcal{P}_i$  for  $i = 1, \dots, n$ . Now let

$$G_i = (n(q-1) + 1)H_i \text{ for } i = 1, \dots, n.$$

and

$$G = G_1 + \dots + G_n.$$

We shall prove, by induction on  $n$ , that the graph  $G$  thus constructed is uniquely  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable.

The result is true for  $n = 1$ .

Now let  $n \geq 2$ . Put

$$V_i = V(G_i), \quad i = 1, \dots, n.$$

Let  $\{W_1, \dots, W_n\}$  be any  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partition of  $G$ . Since  $|V(G_i)| \geq n(q-1) + 1$  for each  $i = 1, \dots, n$ , we have that, for each  $i \in \{1, \dots, n\}$

$$|V_i \cap W_j| \geq q \text{ for at least one } j \in \{1, \dots, n\}.$$



Now suppose two different members of  $\{W_1, \dots, W_n\}$  each contain at least  $q$  vertices of  $V_1$ . Then there are at least  $n+1$  sets of the form  $V_i \cap W_j$  whose cardinality is at least  $q$ . Then, by Dirichlet's principle, there exist integers  $i, r, s \in \{1, \dots, n\}$  with  $r \neq s$  such that

$$|W_i \cap V_r| \geq q \text{ and } |W_i \cap V_s| \geq q,$$

and thus

$$K_{q,q} \subseteq G[W_i].$$

This contradiction proves that only one of the  $W_i$ , say  $W_t$ , contains at least  $q$  vertices of  $V_1$ . Since  $G[V_1]$  contains  $n(q-1)+1$  copies of  $H_1$ , at least one of these copies has all its vertices in  $W_t$ . Our assumption on the ordering of the properties  $\mathcal{P}_1, \dots, \mathcal{P}_n$ , implies that  $H_1 \notin \mathcal{P}_i$  for  $i = 2, \dots, n$  unless  $\mathcal{P}_i = \mathcal{P}_1$ . We may therefore assume, without loss of generality, that  $t = 1$ . Since  $H_1$  is  $\mathcal{P}_1$ -strict, it then follows that

$$W_1 \cap V_i = \emptyset \text{ for } i = 2, \dots, n.$$

By our induction hypothesis, the graph  $G_2 + \dots + G_n$  is uniquely  $(\mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable, so that  $\{V_2, \dots, V_n\}$  is the only  $(\mathcal{P}_2, \dots, \mathcal{P}_n)$ -partition of  $G_2 + \dots + G_n$ . Thus, for each  $i \in \{2, \dots, n\}$ , we have that  $W_i \supseteq V_i$  and hence, since  $G_i$  is  $\mathcal{P}_i$ -strict,  $W_i = V_i$ . This implies that  $\{W_1, \dots, W_n\} = \{V_1, \dots, V_n\}$ , and hence  $G$  is uniquely  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable. ■

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Received 3 January 1997

Revised 25 April 1997