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ON LIGHT SUBGRAPHS IN PLANE GRAPHS OF MINIMUM DEGREE FIVE

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Abstract

A subgraph of a plane graph is light if the sum of the degrees of the vertices of the subgraph in the graph is small. It is well known that a plane graph of minimum degree five contains light edges and light triangles. In this paper we show that every plane graph of minimum degree five contains also light stars $K_{1,3}$ and $K_{1,4}$ and a light 4-path P_4. The results obtained for $K_{1,3}$ and P_4 are best possible.

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1. Introduction

This paper deals only with connected plane graphs which have minimum degree five and which have no face with at most two edges in its boundary. Denote this family of graphs by $\mathcal{G}(5)$.

We use standard terminology and notation of graph theory, see e.g. Bondy and Murty [1]. We recall, however, more specialized notations. Let a *k-vertex* be a vertex of degree k. A path (cycle) on k vertices is defined to be a *k-path* (*k-cycle*, respectively). Let a *k-star* be a star $S_k = K_{1,k} = [X, A_1, A_2, \ldots, A_k]$ on $k+1$ vertices with X as a *central* vertex. A *kleetope* $K(G)$ of a connected plane graph G is defined to be a triangulation obtained from G by placing into each face of G a new vertex and join it to the boundary vertices.

For a subgraph H of a plane graph G the *weight* of $H, w(H)$, is defined to be the sum of degrees of vertices of H in G; namely,

$$w(H) = \sum_{A \in V(H)} \deg_G(A).$$

It is well known that every plane graph contains a vertex of degree at most 5. An excellent Kotzig's Theorem [6,7] states that every 3-connected plane graph contains an edge e with $w(e) \leq 13$; the bound is best possible. These results were further developed in various directions and have served as starting points for discovering many structural properties of plane graphs. Recently, Fabrici and Jendroľ [4] have proved that each 3-connected plane graph having a subgraph isomorphic to a k-path, $k \geq 1$, contains a k-path P_k with all vertices of degrees at most $5k$; this bound is sharp. Moreover, they have shown that for subgraphs other than paths, analogical results cannot be stated, i.e. for any connected plane graph H other than a path and any integer $m \geq 3$ there exists a 3-connected plane graph G in which every subgraph isomorphic to H contains a vertex B such that $\deg_G(B) \geq m$.

On the other hand, Borodin [2] has proved that any $G \in \mathcal{G}(5)$ contains a triangular face τ with $w(\tau) \leq 17$; the bound being precise. (Note that this result has been strengthened in various directions, for a recent progress see Borodin and Saunders [3]).

The above leads to the question, whether there are, in plane graphs of minimum degree five, subgraphs other than paths and triangles with restricted weight.

In this paper we prove that such property has a 3-star (Section 2) and a 4-star (Section 3). For r-stars $r \geq 5$ no similar result can be proved; if we take a graph of the k-sided antiprism, $k \gg r$, insert a new vertex into each its k-gonal face α and join it with all vertices of α, we obtain a *triangulation* (i.e. the plane graph whose all faces are triangles) having only 5-vertices and two k-vertices. In the obtained triangulation every r-star S_r contains a k-vertex.

In Section 4 we prove a tight result for 4-paths with small weight. Section 5 is devoted to the study of light subgraphs in plane triangulations of minimum degree five.

2. Light 3-stars

The main result of this section is as follows.

Theorem 1. *Every planar graph G of minimum degree five contains a star $S_3 = [X, A_1, A_2, A_3]$ such that $\deg_G(X) = 5$ and*

(i) $\deg_G(A_1) = 5$, $\deg_G(A_2) \leq 6$, $\deg_G(A_3) \leq 7$, or

(ii) $\deg_G(A_1) = \deg_G(A_2) = \deg_G(A_3) = 6$.

Moreover, the bounds 6 and 7 are best possible.

Proof. Assume that there exists a counterexample G. Because an insertion of a diagonal into any k-face, $k \geq 4$, of G causes that its endvertices in the resulting graph have degrees at least 6, we can assume that

$$(*) \qquad G \text{ is a triangulation.}$$

According to a consequence of the Euler theorem,

$$(**) \qquad \sum_{A \in V(G)} (\deg_G(A) - 6) = -12.$$

We continue by using the Discharging method. Assign to each vertex $A \in V(G)$ the initial charge $\varphi(A) = \deg_G(A) - 6$. Using the properties of G as a counterexample to our Theorem we define a local redistribution of φ's, preserving their sum such that the new contribution $\widetilde{\varphi}(A)$ is non-negative for all $A \in V$. This will contradict the fact that the sum of the new contribution is, by $(**)$, equal to -12.

For the purposes of this proof, we will call a 5-vertex X to be the *strong* (or *semi-strong*, or *weak*) *neighbour* of a vertex A if there are, in G, two adjacent triangles $[AXY]$ and $[AXZ]$ such that $\deg_G(Y) \geq 6$ and $\deg_G(Z) \geq 6$ (or $\deg_G(Y) = 5$ and $\deg_G(Z) \geq 6$, or $\deg_G(Y) = \deg_G(Z) = 5$, respectively).

The local redistribution is performed according to the following rules.

Rule 1. Each 7-vertex sends $\frac{1}{3}$ to each of its strong neighbours and $\frac{1}{6}$ to each of its semi-strong neighbours.

Rule 2. Each k-vertex, $k \geq 8$ (called the *big* vertex in the sequel) sends $\frac{1}{2}$ to each strong neighbour, $\frac{3}{8}$ to each semi-strong neighbour and $\frac{1}{4}$ to each weak neighbour.

We are going to show that $\widetilde{\varphi}(A) \geq 0$ for each vertex $A \in V(G)$. To this end several cases have to be considered.

(1) Let A be a 5-vertex. Then it has at most two neighbours of degrees ≤ 6.

(1.1) If A is adjacent with at least three big vertices, then A receives from them at least the charge $\frac{1}{4}+2\times\frac{3}{8}$, so $\widetilde{\varphi}(A) \geq -1+\frac{1}{4}+2\times\frac{3}{8} = 0$.

(1.2) A is adjacent with exactly two big vertices. Then at least two from the remaining three neighbours of A are 7-vertices or none of them is a 5-vertex. In the former case $\widetilde{\varphi}(A) \geq -1 + 2 \times \frac{3}{8} + 2 \times \frac{1}{6} > 0$ and in the latter case $\widetilde{\varphi}(A) \geq -1 + 2 \times \frac{1}{2} + \frac{1}{3} > 0$, respectively.

(1.3) A is adjacent with one big vertex. Then either exactly one of the other neighbours is a 5-vertex or at least two are 7-vertices. Hence $\widetilde{\varphi}(A) \geq -1 + \frac{3}{8} + \frac{1}{3} + 2 \times \frac{1}{6} > 0$ or $\widetilde{\varphi}(A) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{3} > 0$, respectively.

(1.4) A is not adjacent with any big vertex. If A is not adjacent with a 5-vertex, we have $\widetilde{\varphi}(A) \geq -1 + 3 \times \frac{1}{3} > 0$. Otherwise, the other vertices in its neighbourhood are 7-vertices and $\widetilde{\varphi}(A) = 2 \times \frac{1}{3} + 2 \times \frac{1}{6} = 0$.

(2) Because any 6-vertex A neither gives nor gets a charge, we have $\widetilde{\varphi}(A) = \varphi(A) = \deg(A) - 6 = 0$.

(3) Let A be a 7-vertex. Because A cannot have weak neighbours it is adjacent to at most four 5-vertices. If A has at most three 5-vertices in the neighbourhood it transfers to them at most the charge $3 \times \frac{1}{3}$, so $\widetilde{\varphi}(A) \geq 1 - 3 \times \frac{1}{3} = 0$. If A is adjacent to exactly four 5-vertices, then at least two of them are semi-strong neighbours of A, thus the charge transferred from A to 5-vertices is at most $2 \times \frac{1}{3} + 2 \times \frac{1}{6} = 1$. Hence in all cases $\widetilde{\varphi}(A) \geq 0$.

(4) Let A be an 8-vertex. According to the value $\frac{1}{2}$ in Rule 2 it is enough to consider only cases of five or more 5-vertices which are adjacent to the vertex A.

(4.1) If eight 5-vertices are adjacent to A, then all are its weak neighbours and, due to Rule 2, we have $\widetilde{\varphi}(A) = 2 - 8 \times \frac{1}{4} = 0$.

(4.2) If seven 5-vertices are adjacent to A, then two of them are semi-strong neighbours and five are weak neighbours, thus $\widetilde{\varphi}(A) = 2 - 2 \times \frac{3}{8} - 5 \times \frac{1}{4} = 0$.

(4.3) Among six 5-vertices adjacent to A there are two semi- strong and four weak neighbours or one strong, two semi-strong and three weak neighbours of four semi-strong and two weak neighbours. This means that $\widetilde{\varphi}(A) = 2 - 2 \times \frac{3}{8} - 4 \times \frac{1}{4} > 0$ or $\widetilde{\varphi}(A) = 2 - \frac{1}{2} - 2 \times \frac{3}{8} - 3 \times \frac{1}{4} = 0$ or $\widetilde{\varphi}(A) = 2 - 4 \times \frac{3}{8} - 2 \times \frac{1}{4} = 0$, respectively.

(4.4) There are five possibilities for a distribution of five 5-vertices in the neighbourhood of the vertex A. One with two strong neighbours, two with exactly one strong neighbour and two with none strong neighbour of A. In all these cases one can easily check that $\widetilde{\varphi}(A) \geq 0$.

(5) Let A be a 9-vertex. Its initial charge is $\varphi(A) = 3$. If A transfers a

charge to at most 6 neighbours then, by Rule 2, the new charge $\widetilde{\varphi}(A) \geq 3 - 6 \times \frac{1}{2} = 0$. If A gives a part of its charge to at least seven 5-valent vertices we can verify analogously as in the case (4) that $\widetilde{\varphi}(A) \geq 0$.We take into consideration the fact that among 5-vertices adjacent to A there are at most one strong neighbour and two or four semi-strong neighbours (in the latter case there is no strong neighbour).

(6) The case when A is an a-vertex, $a \in \{10, 11\}$ is a simple analogue of (4). If A gives a part of its initial charge, according to Rule 2, to at most $2(a-6)$ neighbours, then $\widetilde{\varphi}(A) \geq a - 6 - \frac{1}{2} \times 2(a-6) = 0$. The rest is left to the reader.

(7) Let A be an a-vertex, $a \geq 12$. Then A transfers to each of its neighbours at most $\frac{1}{2}$, hence $\widetilde{\varphi}(A) = \varphi(A) - \frac{1}{2} \times a = a - 6 - \frac{a}{2} = \frac{a-12}{a} \geq 0$.

Thus $\widetilde{\varphi}$ is a non-negative.

Consider the graph obtained by joining two copies of a configuration in Figure 1 via their four half-edges. In this graph, every 5-vertex, except of four ones with four 7-vertices in their neighbourhood, is adjacent to one 5-vertex, one 6-vertex and three 7-vertices.

In the kleetope of the dodecahedron, every 5-vertex is adjacent only to 6-vertices. ■

3. Light 4-stars

Theorem 2. *Each planar graph G of minimum degree 5 contains as a subgraph a star $S_4 = [X, A_1, A_2, A_3, A_4]$ such that $\deg_G(X) = 5$ and $\deg_G(A_i) \leq 10$ for all $i = 1, 2, 3, 4$. Moreover, the bound 10 is best possible.*

Proof. By contradiction. Suppose there is a counterexample, say G, to this theorem. Without loss of generality we can assume that (see the proof of Theorem 1) G is triangulation.

We again use the Discharging method in the sequel. Assign to each vertex A of G the initial charge $\psi(A) = \deg(A) - 6$. Thus, by $(**)$, we have

$$\sum_{A \in V(G)} \psi(A) = -12.$$

Now ψ will be discharged locally according to the following rules preserving the sum of charges equal to -12

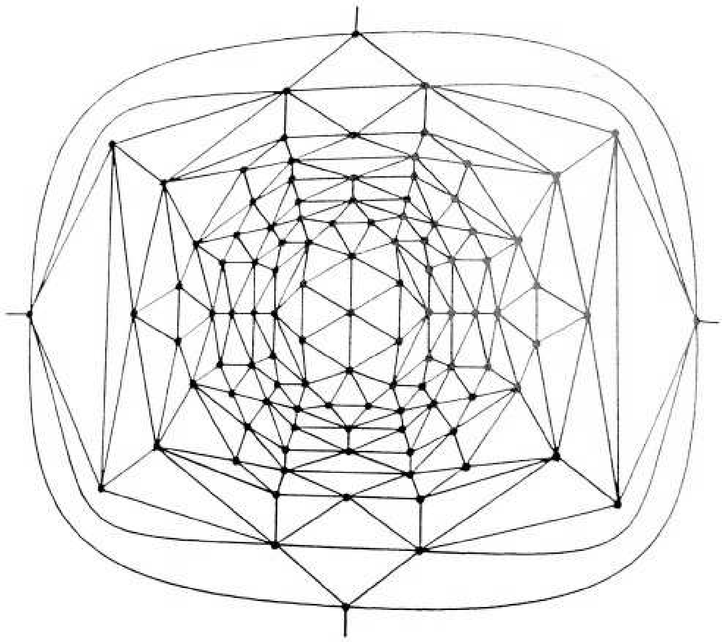

Figure 1

Rule 1. Each k-vertex, $k \geq 12$, sends the charge $\frac{1}{2}$ to each adjacent 5-vertex.

Rule 2. Each 11-vertex A sends the charge $\frac{1}{2}$ to each adjacent 5-vertex X except for the case when X is adjacent to exactly two non-adjacent 5-vertices which are also neighbours of A and the rest two neighbours of X are vertices of degrees ≥ 11. In the exceptional case A sends no charge to X.

We are going to show that after discharging of the charges ψ the new charge function $\widetilde{\psi}$ is non-negative, which provides a contradiction with the negative sum of charges $\widetilde{\psi}'$s. To prove this, several cases have to be considered.

(1) Let A be a 5-vertex. Because G is a counterexample A has at least two neighbours of degree at least 11, which transfer to A the amount of charge $\frac{1}{2}$ each (Note that in the exceptional case, by Rule 2, the mentioned 5-vertex X has three neighbours of degrees ≥ 11 and at least two of them send a charge to X). Hence $\widetilde{\psi}(A) \geq -1 + 2 \times \frac{1}{2} = 0$.

(2) An a-vertex $A, 6 \leq a \leq 10$, neither sends nor receives any charge, therefore $\widetilde{\psi}(A) = \psi(A) = \deg(A) = a - 6 \geq 0$.

(3) Let A be an 11-vertex. If A is adjacent to at most ten 5-vertices, then $\widetilde{\psi}(A) \geq \psi(A) - 10 \times \frac{1}{2} = 11 - 6 - 5 = 0$. Otherwise, there exists a 5-vertex

X in the neighbourhood of A which cannot get, due to Rule 2, any charge from A. Thus A transfers the charge to at most 10 5-vertices and we again have $\widetilde{\psi}(A) \geq 5 - 10 \times \frac{1}{2} = 0$.

(4) Each a-vertex A, $a \geq 12$, can cover requirements of all 5-vertices in its neighbourhood. Thus $\widetilde{\psi}(A) = \psi(A) - a \times \frac{1}{2} = a - 6 - \frac{a}{2} = \frac{a-12}{2} \geq 0$.

Thus $\widetilde{\psi}$ is a non-negative function.

Consider the graph of an Archimedian polytope $(3, 10, 10)$ (it can be obtained from the dodecahedron by cutting its vertices). Into each its 10-face insert a new vertex and join it to the boundary vertices. In the graph thus obtained, each star S_4 with central 5-vertex contains a vertex of degree 10. So the bound 10 is best possible. ■

4. LIGHT PATHS

Already Wernicke [8], trying to prove the Four Colour Theorem, observed that each plane graph $G \in \mathcal{G}(5)$, of minimum degree 5, contains an edge e (i.e. a 2-path P_2) such that $w(P_2) \leq 11$. Franklin [5] extended this result to 3-paths, he proved that every graph $G \in \mathcal{G}(5)$ contains a 3-path P_3 with $w(P_3) \leq 17$, the bound being best possible.

In [4] it is proved that each plane 3-connected graph containing a k-path, $k \geq 1$, has a subgraph isomorphic with a k-path P_k such that $w(P_k) \leq 5k^2$.

Problem 1. What is the best upper bound on $w(P_k)$ if we restrict ourselves to the graphs from the family $\mathcal{G}(5)$?

Here we provide the answer for the case $k = 4$, namely we have

Theorem 3. *Each connected planar graph of minimum degree* 5 *contains a path* P_4 *such that*

$$w(P_4) \leq 23.$$

Moreover, the bound 23 *is best possible.*

Proof. By contradiction. Suppose G is a counterexample to our theorem having n vertices and a maximum number of edges among all counterexamples on n vertices.

Proposition 1. *G is a triangulation.*

Proof. Assume G contains a k-face α, $k \geq 4$. Then α is incident with two nonadjacent vertices X and Y such that $\deg_G(X) + \deg_G(Y) \geq 12$, otherwise there is a 4-path Q on the boundary of α with $w(Q) \leq 23$. If we insert a diagonal XY into α, in the obtained graph $\widetilde{G}$ there is $\deg_{\widetilde{G}}(X) + \deg_{\widetilde{G}}(Y) \geq 14$. Because any other vertex of $\widetilde{G}$ is of degree at least 5, the edge XY cannot appear in any 4-path R with $w(R) \leq 23$. ∎

Proposition 2. *Every a-vertex, $a \in \{7, 8\}$, is adjacent to at most $\lfloor \frac{a}{2} \rfloor$ 5-vertices.*

Proof. In the oposite case, due to Proposition 1, we can find a 4-path P_4 with $w(P_4) \leq 23$, a contradiction. ∎

In the rest of the proof we again use the Discharging method. Assign to each vertex $A \in V(G)$ the initial charge $g(A) = \deg_G(A) - 6$. Therefore $(**)$ is equivalent to

$$\sum_{A \in V(G)} g(A) = -12.$$

We use the following rule in order to transform g into a new charge function $h : V \to \mathbb{Q}$ by redistributing charges locally so that the sum of new charges remains the same.

Rule. Each k-vertex $A, k \geq 6$, transfers the charge $\frac{\deg(A)-6}{m(A)}$ to each adjacent 5-vertex; $m(A)$ denotes the number of 5-vertices adjacent to A. If $m(A) = 0$, no charge is transferred.

We are going to show that h is a non-negative function, which will trivially be a contradiction because

$$\sum_{A \in V(G)} g(A) = \sum_{A \in V(G)} h(A) = -12.$$

Case 1. A is a k-vertex, $k \geq 6$. The A vertex transfers, by the Rule, the charge $\frac{\deg(A)-6}{m(A)}$ to $m(A)$ 5-vertices in its neighbourhood, so $h(A) = g(A) - m(A) \times \frac{\deg(A)-6}{m(A)} = 0$ if $m(A) \neq 0$, or $h(A) = g(A) \geq 0$ if $m(A) = 0$.

Case 2. A is a 5-vertex. Then A is adjacent to at least three vertices of degrees at least 7, otherwise it is adjacent to at least three vertices of degrees at most 6 and one can easily find a 4-path P_4 with $w(P_4) \leq 5 + 3 \times 6 = 23$. By the rule each neighbour of A having degree at least 7 transfers to A the charge at least $\frac{1}{3}$. We have $h(A) \geq -1 + 3 \times \frac{1}{3} = 0$.

Thus h is a non-negative function.

In the kleetope of the graph of Figure 2 every path P_4 has weight $w(P_4) \geq 23$. Thus the bound 23 is best possible. ∎

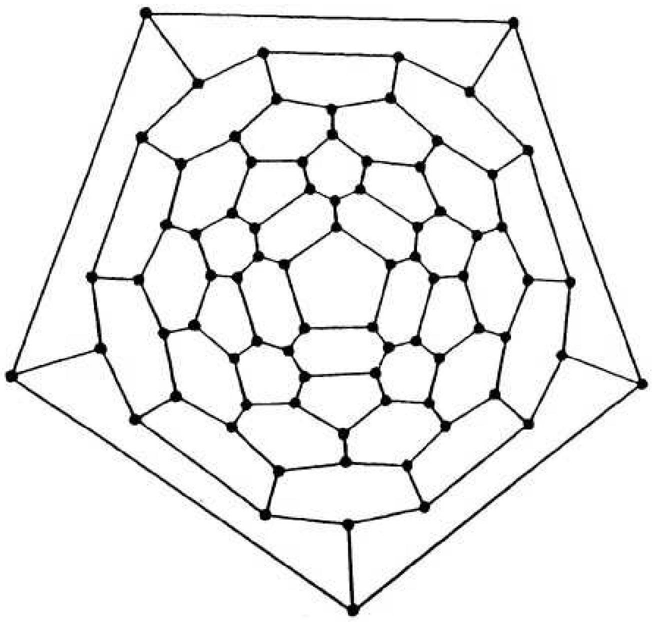

Figure 2

5. Triangulations of Minimum Degree Five

Theorems 1 and 2 immediately provide the following

Corollary 4. *Each plane triangulation of minimum degree* 5 *contains*

(i) *a* 3*-cycle* C_3 *with* $w(C_3) \leq 18$,
(ii) *a* 4*-cycle* C_4 *with* $w(C_4) \leq 35$ *and*
(iii) *a* 5*-cycle* C_5 *with* $w(C_5) \leq 45$.

■

We believe that the bounds in (ii) and (iii) can be sufficiently improved. As we mentioned above, by Borodin [2], the best possible bound for $w(C_3)$ is 17.

Problem 2. Is there a constant $\sigma(k)$ such that every plane triangulation of minimum degree 5, which contains a k-cycle, contains a k-cycle C_k with $w(C_k) \leq \sigma(k)$?

The following is an analogue of Theorems 1 and 2.

Theorem 5. *Every plane triangulation of minimum degree* 5 *contains*

(i) *a* 3*-star* S_3 *in which the central vertex has degree at most* 8 *and other vertices are of degree at most* 6

(ii) *a* 4*-star* S_4 *in which the central vertex has degree at most* 11 *and other vertices are of degrees at most* 7*.*

Proof is left to the reader. It is enough to use the Discharging method with the following rule for redistributing charges

Rule. Each k-vertex $A, k \geq 7$, transfers to each adjacent 5-vertex the charge $\frac{\deg(A)-6}{m(A)}$, where $m(A)$ is the number of 5-vertices adjacent to A. If $m(A) = 0$, A transfers no charge. ∎

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