

## ***KP*-DIGRAPHS AND *CKI*-DIGRAPHS SATISFYING THE $k$ -MEYNIEL'S CONDITION**

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### **Abstract**

A digraph  $D$  is said to satisfy the  $k$ -Meyniel's condition if each odd directed cycle of  $D$  has at least  $k$  diagonals.

The study of the  $k$ -Meyniel's condition has been a source of many interesting problems, questions and results in the development of Kernel Theory.

In this paper we present a method to construct a large variety of kernel-perfect (resp. critical kernel-imperfect) digraphs which satisfy the  $k$ -Meyniel's condition.

**Primary keywords:** digraph, kernel, independent set of vertices, absorbing set of vertices, kernel-perfect digraph, critical-kernel-imperfect digraph,  $\tau$ -system,  $\tau_1$ -system.

**Secondary keywords:** indepedent kernel modulo  $Q$ , co-rooted tree,  $\tau$ -construction,  $\tau_1$ -construction.

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### 1. INTRODUCTION

For general concepts we refer the reader to [1]. If  $D$  is a digraph, then  $V(D)$  and  $FD$  or  $F(D)$  will denote the sets of vertices and arcs of  $D$  respectively. We write  $D_0 \subseteq D$  (resp:  $D_0 \subseteq^* D$ ) whenever  $D_0$  is a subdigraph (resp: induced subdigraph) of  $D$ . For  $S_1, S_2 \subset V(D)$ , the arc  $u_1 u_2$  of  $D$  will be called an  $S_1 S_2$ -arc provided that  $u_1 \in S_1$  and  $u_2 \in S_2$ ;  $D[S_1]$  will denote the subdigraph of  $D$  induced by  $S_1$  and  $D[S_1, S_2]$  the subdigraph of  $D$  with vertex set  $S_1 \cup S_2$  whose arcs are the  $S_1 S_2$ -arcs of  $D$ . The *asymmetrical part* of  $D$  (resp: *symmetrical part* of  $D$ ), which is denoted by  $Asym D$  (resp:

$Sym D$ ) is the spanning subdigraph of  $D$  whose arcs are the asymmetrical (resp: symmetrical) arcs of  $D$ .

The set  $I \subset V(D)$  is *independent* if  $FD[I] = \emptyset$ . A *kernel*  $N$  of  $D$  is an independent set of vertices such that for every  $z \in (V(D) - N)$  there exists a  $zN$ -arc in  $D$ . A *semikernel*  $S$  of  $D$  is an independent set of vertices such that for every  $z \in (V(D) - S)$  for which there exists an  $Sz$ -arc, there also exists a  $zS$ -arc.

A digraph  $D$  is called

- (i) *quasi KP-digraph* if every proper induced subdigraph of  $D$  has a kernel,
- (ii) *kernel-perfect digraph* or *KP-digraph* if every induced subdigraph of  $D$  has a kernel,
- (iii) *critical kernel-imperfect* or *CKI-digraph* if  $D$  is a quasi *KP*-digraph and has no kernel.

It was proved by Neumann-Lara in [9] that  $D$  is a *KP*-digraph iff every induced subdigraph of  $D$  has a non empty semikernel. We will say that a digraph  $A$  is a *co-rooted tree* if  $A$  is an asymmetrical digraph whose underlying graph is a tree and there exists one and only one vertex  $v \in F(A)$  (the *co-root* of  $A$ ) such that there is no arc in  $A$  whose initial endvertex is  $v$ .

Let  $C = (1, 2, \dots, m, 1)$  be a directed cycle of  $D$ , we denote by  $\ell(C)$  its length, for  $i \neq j$   $i, j \in V(C)$  we denote by  $(i, C, j)$  the  $ij$ -directed path contained in  $C$  and we denote by  $\ell(i, C, j)$  its length; an arc  $f = ij \in (FD - FC)$  is a *diagonal* of  $C$  iff  $i \neq j$ ,  $i, j \in V(C)$  and  $\ell(i, C, j) < \ell(C) - 1$  and  $f$  is a *pseudodiagonal* when  $\ell(i, C, j) \leq \ell(C) - 1$ .

A digraph  $D$  is said to satisfy the  $k$ -Meyniel's condition if each odd directed cycle of  $D$  has at least  $k$  diagonals.

The study of the  $k$ -Meyniel's condition has been a source of many interesting problems, questions and results in the development of Kernel Theory (see by example [2], [3], [4], [5], [6]).

In this paper we present a method to construct a large variety of kernel-perfect (resp. critical kernel-imperfect) digraphs which satisfy the  $k$ -Meyniel's condition. This method is also the basis in the study of extensions of kernel-perfect digraphs to critical kernel-imperfect digraphs (see [8]).

**Theorem 1.1** [8]. *Let  $D_1, D_2$  and  $D$  be digraphs such that  $V(D_1) \cap V(D_2) = \{v\}$  and  $D = D_1 \cup D_2$ . Then  $D$  is a *KP*-digraph iff  $D_1$  and  $D_2$  are *KP*-digraphs.*

**Theorem 1.2.** *Let  $G$  be a connected graph without cycles and for each  $e = w_1 w_2 \in E(G)$  let  $\gamma_e$  be a digraph such that  $\{w_1, w_2\} \subseteq V(\gamma_e)$ ,  $V(\gamma_e) \cap V(G) = \{w_1, w_2\}$ . Suppose that the digraphs  $(\gamma_e - V(G))_{e \in E(G)}$  are mutually disjoint. The digraph  $D = \bigcup_{e \in E(G)} \gamma_e$  is a KP-digraph iff  $\gamma_e$  is a KP-digraph for each  $e \in E(G)$ .*

**Proof.** Theorem 1.2 follows directly from Theorem 1.1 proceeding by induction on  $|V(G)|$ . ■

**Theorem 1.3** [6]. *Suppose that  $V(D)$  has a partition  $\{V_1, V_2\}$  such that every  $V_1 V_2$ -arc in  $D$  is symmetric and  $D[V_1]$  and  $D[V_2]$  are KP-digraphs. Then  $D$  is a KP-digraph.*

**Theorem 1.4** [6]. *If  $D$  is a CKI-digraph, there is no a partition  $\{V_1, V_2\}$  of  $V(D)$  such that  $D[V_1, V_2] \subseteq \text{Sym } D$ ; in other words  $\text{Asym } D$  is strongly connected.*

## 2. $\tau_1$ -SYSTEM AND $\tau_1$ -CONSTRUCTION

**Definition 2.1.** Let  $D$  be a multidigraph and  $u \in V(D)$ ; a partition  $\pi_u = \{u_-^0, u_-^1, \dots, u_-^{m(u)-1}, u_+\}$  of  $F_u(D) = F_u^+(D) \cup F_u^-(D)$  will be called a  $\tau$ -partition in  $u$  if it satisfies the following two properties:

- (1)  $u_-^i \subseteq F_u^-(D)$  for each  $i \in \{0, 1, \dots, m(u) - 1\}$ .
- (2)  $u_+ = F_u^+(D)$ .

$F_u^+(D)$  (resp:  $F_u^-(D)$ ) denotes the set of arcs of  $D$  whose initial (resp: terminal) endvertex is  $u$ .

When  $\pi_u$  is a  $\tau$ -partition in  $u$  we denote by  $\bar{\pi}_u$  the set

$$\bar{\pi}_u = \left\{ (u, u_+), (u, u_-^i) \mid i \in \{0, 1, \dots, m(u) - 1\} \right\}.$$

**Definition 2.2.** A triple  $t_0 = (D_0, U, A)$  will be called a  $\tau_0$ -system if it satisfies the following two properties:

- (1)  $D_0$  is a multidigraph,  $U \subseteq V(D_0)$ .
- (2)  $A = (A_u)_{u \in U}$  is a family of co-rooted trees with  $V(A_u) = \bar{\pi}_u$  where  $\pi_u$  is a  $\tau$ -partition in  $u$ ,  $(u, u_+)$  is the co-root of  $A_u$  and  $|\pi_u| \geq 2$ .

For each  $u \in U$  and  $f \in F_u(D)$  we denote by  $\pi_u(f)$  the element of  $\pi_u$  containing  $f$ .

If  $t_0 = (D_0, U, A)$  is a  $\tau_0$ -system, then  $\tau_0(t_0)$  denotes the digraph defined as follows:

$$\begin{aligned} V(\tau_0(t_0)) &= V(D_0 - U) \cup \bigcup_{u \in U} V(A_u), \\ F(\tau_0(t_0)) &= \{ f^* \mid f \in F D_0 \} \end{aligned}$$

for each  $f = w z \in F D_0$ ,  $f^*$  is defined by

$$f^* = \begin{cases} f, & \text{when } \{ w, z \} \subseteq (V(D_0) - U), \\ w(z, \pi_z(f)), & \text{when } w \in (V(D_0) - U) \text{ and } z \in U, \\ (w, w_+)z, & \text{when } w \in U \text{ and } z \in (V(D_0) - U), \\ (w, w_+)(z, \pi_z(f)), & \text{when } \{ w, z \} \subseteq U. \end{cases}$$

**Definition 2.3.** A pair  $t_1 = (t_0, \gamma)$  will be called a  $\tau_1$ -system if  $t_0 = (D_0, U, A)$  is a  $\tau_0$ -system and  $\gamma = (\gamma_u)_{u \in U}$  is a family, where  $\gamma_u = (\gamma_u^f)_{f \in F(A_u)}$  is a family of internally disjoint directed paths. Moreover, if  $f = w_1 w_2$ , then  $\gamma_u^f$  is a  $w_1 w_2$ -directed path of positive even length and  $V(\gamma_u^f) \cap V(A_u) = \{ w_1, w_2 \}$ . Also we denote  $t_1 = (D_0, U, A, \gamma)$ .

Note that  $V(\gamma_{u_1}^{f_1}) \cap V(\gamma_{u_2}^{f_2}) = \emptyset$  for any  $f_1 \in F A_{u_1}$ ,  $f_2 \in F A_{u_2}$  and  $u_1 \neq u_2$ .

If  $t_1 = (t_0, \gamma)$  is a  $\tau_1$ -system, then we denote  $\tau_1(t_1) = \tau_0(t_0) \cup \bigcup_{u \in U} \bigcup_{f \in F A_u} \gamma_u^f$ .

**Definition 2.4** [5]. If  $D$  is a digraph and  $N, Q \subset V(D)$ ,  $N^c = V(D) - N$ ,  $Q^c = V(D) - Q$ ,  $N$  is said to be an independent kernel *modulo*  $Q$  (i.k. mod  $Q$ ) of  $D$  iff

- (i)  $N$  is independent,
- (ii) For every  $w \in N^c \cap Q^c$  there exists a  $w N$ -arc.

**Observation 2.1.** If  $D$  is a directed path of positive even length say  $D = (u_0, u_1, \dots, u_{2n})$ ,  $n \geq 1$ , then  $D$  satisfies the following properties:

- (i) If  $N$  is an i.k. mod  $\{u_{2n}\}$  of  $D$ , then  $u_0 \in N$  iff  $u_{2n} \in N$ .
- (ii)  $\{u_i \mid i = 2k, 0 \leq k \leq n\}$  is an i.k. mod  $\{u_{2n}\}$ , in fact it is a kernel of  $D$  which contains  $\{u_0, u_{2n}\}$ .
- (iii)  $N = \{u_{2i+1} \mid 0 \leq i \leq n-1\}$  is an i.k. mod  $\{u_{2n}\}$  of  $D$  such that  $\{u_0, u_{2n}\} \subseteq N^c$ .

**Definition 2.5.** Let  $D$  be a multidigraph,  $R, T \subseteq V(D)$ ;  $T$  will be called  $R$ -homogeneous whenever  $T \subseteq R$  or  $T \subseteq (V(D) - R)$ .

**Lemma 2.1.** Let  $A$  be a co-rooted tree with co-root  $a_0$ ,  $|V(A)| \geq 2$  and  $(\gamma^f)_{f=(u_f, v_f) \in FA}$  a family of internally disjoint directed paths of positive even length such that  $\gamma^f$  is a  $u_f v_f$ -directed path and  $V(\gamma^f) \cap V(A) = \{u_f, v_f\}$ . If  $N$  is an i.k. mod  $\{a_0\}$  of  $\bigcup_{f \in FA} \gamma^f$ , then  $V(A)$  is  $N$ -homogeneous. Moreover, when  $V(A) \subseteq N^c$  there is no  $a_0$   $N$ -arc in  $\bigcup_{f \in FA} \gamma^f$ .

**Proof.** The proof is by induction on  $|V(A)|$ . If  $|V(A)| = 2$  the result is a directed consequence of Observation 2.1 (i). Suppose that  $|V(A)| > 2$  and let  $g = uw \in F(A)$  be an arc such that  $\delta_A^-(u) = 0$ ,  $N$  be an i.k. mod  $\{a_0\}$  of  $\bigcup_{f \in FA} \gamma^f$  and  $A_0 = A - \{u\}$ . Clearly we have:

- (1)  $N \cap \bigcup_{f \in FA_0} V(\gamma^f)$  is an i.k. mod  $\{a_0\}$  of  $\bigcup_{f \in FA_0} \gamma^f$  (because  $\delta_A^-(u) = 0$ ) thus by the inductive hypothesis  $V(A_0)$  is  $N$ -homogeneous.
- (2)  $N \cap V(\gamma^g)$  is an i.k. mod  $\{w\}$  of  $\gamma^g$  and Observation 2.1 (i) implies  $\{u, w\}$  is  $N$ -homogeneous.

It follows from (1) and (2) that  $V(A)$  is  $N$ -homogeneous. When  $V(A) \subseteq N^c$  it follows from the choice of  $a_0$  that there is no  $a_0 \bigcup_{f \in FA} V(\gamma^f)$ -arc, so there is no  $a_0$   $N$ -arc in  $\bigcup_{f \in FA} \gamma^f$ . ■

**Theorem 2.1.** Let  $t_1 = (D_0, U, A, \gamma)$  be a  $\tau_1$ -system. If  $D_0$  has a kernel, then  $D = \tau_1(t_1)$  has a kernel.

**Proof.** Let  $N_0$  be a kernel of  $D_0$ , Observation 2.1 implies that for each  $u \in U$  and  $f = w_1 w_2 \in F(A_u)$  there exist  $N_{u,f}^i$ ,  $i \in \{0, 1\}$  independent kernels mod  $\{w_2\}$  of  $\gamma_u^f$  such that  $\{w_1, w_2\} \subseteq N_{u,f}^0$  and  $\{w_1, w_2\} \subseteq (N_{u,f}^1)^c$ . It is easy to see by using Lemma 2.1 that

$$\begin{aligned} N = & [N_0 \cap (V(D_0) - U)] \cup \left( \bigcup_{u \in N_0 \cap U} \bigcup_{f \in FA_u} N_{u,f}^0 \right) \\ & \cup \left( \bigcup_{u \in N_0 \cap U^c} \bigcup_{f \in F(A_u)} N_{u,f}^1 \right) \end{aligned}$$

is a kernel of  $\tau_1(t_1)$ . ■

**Theorem 2.2.** *Let  $t_1 = (D_0, U, A, \gamma)$  be a  $\tau_1$ -system. If  $D = \tau_1(t_1)$  has a kernel, then  $D_0$  has a kernel.*

**Proof.** Let  $N$  be a kernel of  $D$ ; it is easy to see that for each  $u \in U$ ,  $N \cap (\bigcup_{f \in FA_u} V(\gamma_u^f))$  is an i.k. mod  $\{(u, u_+)\}$  of  $\bigcup_{f \in FA_u} \gamma_u^f$  and Lemma 2.1 implies  $V(A_u)$  is  $N \cap (\bigcup_{f \in FA_u} V(\gamma_u^f))$ -homogeneous and hence  $V(A_u)$  is  $N$ -homogeneous and when  $V(A_u) \subseteq N^c$  there is no

$$(u, u_+) \left[ N \cap \left( \bigcup_{f \in FA_u} V(\gamma_u^f) \right) \right] \text{-arc}$$

in  $D$  and it follows that

$$N_0 = \left[ N \cap (V(D) - \bigcup_{u \in U} \bigcup_{f \in FA_u} V(\gamma_u^f)) \right] \cup \{ u \in U \mid V(A_u) \subseteq N \}$$

is a kernel of  $D_0$ . ■

**Definition 2.6.** Let  $A$  be a co-rooted tree, a subset  $S$  of  $V(A)$  will be called an *initial section* of  $A$  if for each  $w \in A$  such that there exists a  $wS$ -directed path in  $A$ , we have  $w \in S$ .

Clearly the empty set is an initial section of any co-rooted tree.

**Theorem 2.3.** *Let  $t_1 = (D_0, U, A, \gamma)$  be a  $\tau_1$ -system. Suppose that for each non trivial family  $S = (S_u)_{u \in U}$ , where  $S_u$  is an initial section of  $A_u$ , the digraph  $D_0 - \bigcup_{u \in U} \{f \in FD_0 \mid f^* \text{ incides in } S_u\}$  is a KP-digraph (for each  $f \in FD_0$ ,  $f^*$  denotes the arc of  $\tau_0(t_0)$  defined as in Definition 2.2). If every proper induced subdigraph of  $D_0$  has a kernel, then every proper induced subdigraph of  $D = \tau_1(t_1)$  has a kernel.*

**Proof.** First we recall that if  $G$  and  $H$  are digraphs then  $G \cap H$  denote the digraph whose vertex set is  $V(G) \cap V(H)$  and  $A(G \cap H) = A(G) \cap A(H)$ . Now, if Theorem 2.3 were false,  $D$  would contain a proper induced *CKI*-subdigraph. Let  $H$  be a proper induced *CKI*-subdigraph of  $D$ . First we will prove that for each  $u \in U$ ,

$$H \cap D \left[ \bigcup_{f \in FA_u} V(\gamma_u^f) \right] = D \left[ \bigcup_{f \in F(A_u - S_u)} V(\gamma_u^f) \right],$$

where  $S = (S_u = V(A_u) - V(H))_{u \in U}$  is a family such that  $S_u$  is an initial section of  $A_u$ .

Let  $u \in U$ , when  $H \cap D \left[ \bigcup_{f \in FA_u} V(\gamma_u^f) \right] = D \left[ \bigcup_{f \in FA_u} V(\gamma_u^f) \right]$ , then  $S_u = \emptyset$  satisfies the required properties. If

$$H \cap D \left[ \bigcup_{f \in FA_u} V(\gamma_u^f) \right] \subsetneq D \left[ \bigcup_{f \in FA_u} V(\gamma_u^f) \right]$$

since

$$H \cap D \left[ \bigcup_{f \in FA_u} V(\gamma_u^f) \right] = \bigcup_{f \in FA_u} H \cap D [V(\gamma_u^f)]$$

then there exists  $f = w_1 w_2 \in FA_u$  such that  $H \cap D [V(\gamma_u^f)] \subsetneq D [V(\gamma_u^f)] = \gamma_u^f$ . Since *Asym*  $H$  is strongly connected (see Theorem 1.3) Definition 2.3 implies

$$H \cap D [V(\gamma_u^f)] \subseteq D [\{w_1, w_2\}] ;$$

now we consider  $A_u^{w_1} = A_u [\{z \in V(A_u) \mid \text{there exists a } zw_1\text{-directed path contained in } A_u\}]$  and

$$H_{w_1} = H \cap D \left[ \bigcup_{f \in FA_u^{w_1}} V(\gamma_u^f) \right] .$$

So, we have that  $H_{w_1} = \emptyset$  since, if  $H_{w_1} \neq \emptyset$  then  $H_2 = H [V(H) - V(H_{w_1})]$  is a *KP*-digraph such that  $H_2 \cap D [V(\gamma_u^f)] \subseteq D [\{w_2\}]$  (since  $H \cap D [V(\gamma_u^f)] \subseteq D [\{w_1, w_2\}]$ ). Furthermore, since for each  $f \in FA_u$ ,  $\gamma_u^f$  is a *KP*-digraph and,  $A_u$  is a co-rooted tree, Theorem 1.2 implies that  $\bigcup_{f \in FA_u} \gamma_u^f$

is a *KP*-digraph and clearly  $H_{w_1} \subseteq^* \left( \bigcup_{f \in FA_u} \gamma_u^f \right)$  so  $H_{w_1}$  is a *KP*-digraph;

Definition 2.3 and  $H_2 \cap D [V(\gamma_u^f)] \subseteq \{w_2\}$  imply there is no  $H_{w_1} H_2$ -arcs in  $H$  and using Theorem 1.3 we conclude that  $H$  is a *KP*-digraph which is impossible. So, we have proved that  $H_{w_1} = \emptyset$  and then

$$H \cap D \left[ \bigcup_{f \in FA_u} V(\gamma_u^f) \right] \subseteq^* D \left[ \bigcup_{f \in F(A_u - A_u^{w_1})} V(\gamma_u^f) \right]$$

for each  $f \in FA_u$  such that

$$H \cap D \left[ V(\gamma_u^f) \right] \subsetneq D \left[ V(\gamma_u^f) \right],$$

and this implies

$$H \cap D \left[ \bigcup_{f \in FA_u} V(\gamma_u^f) \right] = D \left[ \bigcup_{f \in F(A_u - S_u)} V(\gamma_u^f) \right],$$

where  $S_u = S_u^1 \cup S_u^2$ ,  $S_u^1 = \{w \in V(A_u) \mid \text{there exists } f = wz \in FA_u \text{ such that}$

$$H \cap D \left[ V(\gamma_u^f) \right] \subsetneq \gamma_u^f\}$$

and  $S_u^2 = \bigcup_{w \in S_u^1} V(A_u^w)$ . Clearly,  $S_u$  is an initial section of  $A_u$ .

Let  $H_0$  be a subdigraph (not necessarily induced) of  $D_0$  obtained from  $H$  by identifying  $\bigcup_{f \in FA_u} V(\gamma_u^f)$  with  $u$ , for each  $u \in U$  such that  $H \cap$

$D \left[ \bigcup_{f \in FA_u} V(\gamma_u^f) \right] \neq \emptyset$ . So, we have that  $H \cong \tau_1(H_0, U_0, (A_u - S_u)_{u \in U_0}, \gamma_0)$  where,  $U_0 = U \cap V(H_0)$  and  $\gamma_0$  is the restriction of  $\gamma$  to  $\bigcup_{u \in U} F(A_u - S_u)$ .

Now we will prove that  $H_0$  has a kernel.

If  $S_u = \emptyset$  for each  $u \in U$ , then there exists

$$z \in \left( V(D) - \bigcup_{u \in U} \bigcup_{f \in FA_u} V(\gamma_u^f) \cap (V(D) - V(H)) \right)$$

and hence  $H_0$  is a proper induced subdigraph of  $D_0$  and the hypothesis implies  $H_0$  has a kernel.

If  $S_u \neq \emptyset$  for some  $u \in U$ , then  $H_0$  is an induced subdigraph of  $D_0 - \bigcup_{u \in U'} \{ f \in FD_0 \mid f^* \text{ incides in } S_u \}$ , where  $U' = \{ u \in U \mid S_u \neq \emptyset \}$  ( $f^*$  is defined as in Definition 2.2) and the hypothesis implies that  $H_0$  has a kernel.

Since  $H_0$  has a kernel and  $H = \tau_1(H_0, U_0, (A_u - S_u)_{u \in U_0}, \gamma_0)$ , it follows from Theorem 3.1 that  $H$  has a kernel contradicting that  $H$  is a *CKI*-digraph.  $\blacksquare$

**Theorem 2.4.** *Let  $t_1 = (D_0, U, A, \gamma)$  be a  $\tau_1$ -system. If every proper induced subdigraph of  $D = \tau_1(t_1)$  has a kernel, then every proper induced subdigraph of  $D_0$  has a kernel.*



**Proof.** Let  $D'_0$  be a proper induced subdigraph of  $D_0$  and

$$D' = \tau_1(D'_0, U', (A_u)_{u \in U'}, \gamma'),$$

where  $U' = (U \cap V(D'_0))$  and  $\gamma'$  is the restriction of  $\gamma$  to  $\bigcup_{u \in U'} FA_u$ ; since  $D'$  is a proper induced subdigraph of  $D$ , we have that  $D'$  has a kernel and Theorem 2.2 implies that  $D'_0$  has a kernel. ■

**Theorem 2.5.** *Let  $t_1 = (D_0, U, A, \gamma)$  be a  $\tau_1$ -system such that for every non trivial family  $S = (S_u)_{u \in U}$ , where  $S_u$  is an initial section of  $A_u$ , the digraph  $D_0 - \bigcup_{u \in U} \{f \in FD_0 \mid f^* \text{ incides in } S_u\}$  is a KP-digraph. Then  $\tau_1(t_1)$  is a KP-digraph (resp: CKI-digraph) if and only if  $D_0$  is a KP-digraph (resp: CKI-digraph).*

### 3. $\tau_1$ -CONSTRUCTIONS

In these section we present a method to realize in a simple way some  $\tau_1$ -constructions and we obtain a large variety of KP digraphs and CKI-digraphs satisfying the  $k$ -Meyniel's condition.

Let  $D_0$  be a multidigraph,  $U \subseteq V(D_0)$ ,  $<^p$  be a total order in  $\{v(f) = \{u_1, u_2\} \mid f \text{ is an } u_1 u_2\text{-arc}\}$ , and  $<^{u_1 u_2}$  be a total order in  $\{f \in FD_0 \mid f \text{ is an } u_1 u_2\text{-arc}\}$ . We will denote by  $<$  the total order defined in

$$\bigcup_{u \in U} \{(v(f), f) \mid f \in F(\text{Sym } D_0) \cap F_u^-(D_0)\}$$

as follows:  $(v(f), f) < (v(g), g)$  if and only if  $v(f) <^p v(g)$  or  $v(f) = v(g) = \{u_1, u_2\}$  and  $f <^{u_1 u_2} g$ . And for each  $u \in U$  we will denote by  $u_-(f) = \{f\}$  when  $f \in F(\text{Sym } D_0) \cap F_u^-(D_0)$ ;  $u_-^0 = F(\text{Asym } D_0) \cap F_u^-(D_0)$ ,  $u_+ = F_u^+(D_0)$ ,

$$\Pi_u = \{u_+, u_-^0, u_-(f) \mid f \in F(\text{Sym } D_0) \cap F_u^-(D_0)\}.$$

(clearly  $\Pi_u$  is a  $\tau$ -partition in  $u$ ),  $A_u^<$  the  $u_-^0 u_+$ -directed path defined as follows

$$A_u^< = (u_-^0, u_-(f_1), u_-(f_2), \dots, u_-(f_r), u_+),$$

where

$$(v(f_1), f_1) < (v(f_2), f_2) < \dots < (v(f_r), f_r)$$

and

$$\{f_1, \dots, f_r\} = F \text{Sym } D_0 \cap F_u^- D_0.$$

Finally we denote by  $A^< = (A_u^<)_{u \in U}$ .

**Theorem 3.1.** *Let  $D_0$  be a multidigraph which is a quasi  $KP$ -digraph and  $t_0 = (D_0, U, A^<)$  any  $\tau_0$ -system defined as at the begining of this section. For any non trivial family  $S = (S_u)_{u \in U}$ , where  $S_u$  is an initial section of  $A_u^<$ ,  $(D_0 - \bigcup_{u \in U} \{f \in FD_0 \mid f^* \text{ incides in } S_u\}) = D_0(S)$  is a  $KP$ -digraph.*

**Proof.** Suppose that there exists a non trivial family  $S = (S_u)_{u \in U}$ , where  $S_u$  is an initial section of  $A_u^<$ , such that  $D_0(S)$  is not a  $KP$ -digraph and let  $D_1$  be a  $CKI$ -digraph which is an induced subdigraph of  $D_0(S)$ ; since  $D_0(S)$  is a proper subdigraph of  $D_0$ , we have that  $D_1$  is not an induced subdigraph of  $D_0$  and there exists an  $uv$ -arc in  $FD_0[V(D_1)] - F(D_1)$  and  $S_v$  is not empty, so  $v_-^0 \cap FD_1 = \emptyset$ . Since  $D_1$  is a  $CKI$ -digraph, Theorem 1.4 implies that there exists some  $wv$ -arc in  $Asym D_1$  and  $\emptyset = v_-^0 \cap FD_1 = F Asym D_0 \cap F_v^-(D_0) \cap FD_1$  implies  $wv \in F(\text{Sym } D_0)$ , so there exists  $w \in V(D_1)$  such that  $\delta_{D'_1}^+(w) \neq 0$ , where  $D'_1 = Asym D_1 \cap \text{Sym } D_0$ . Futhermore, if  $zv \in F_z^+(D'_1)$ , then  $S_z \neq \emptyset$ ,  $z_-^0 \cap F(D_1) = \emptyset$  and since  $D_1$  is a  $CKI$ -digraph, Theorem 1.4 implies  $F(Asym D_1) \cap F_z^-(D'_1) \neq \emptyset$  and then there exists  $wz \in F(D'_1)$ ; hence  $\delta_{D'_1}^+(w) \neq 0$ . We have proved:

(a) there exists  $w \in V(D'_1)$  such that  $\delta_{D'_1}^+(w) \neq 0$ .

(b) if  $\delta_{D'_1}^+(z) \neq 0$ , then  $\delta_{D'_1}^-(z) \neq 0$ .

It follows that  $D'_1$  contains a directed cycle  $\mathcal{C} = (w_0, f_0, w_1, f_1, \dots, w_n, f_n, w_0)$  where  $\{w_0, \dots, w_n\} \subseteq V(D'_1)$ ,  $\{f_0, \dots, f_n\} \subseteq FD'_1$ . Since  $<^p$  is a total order in  $\{v(f) \mid f \in \text{Sym } D_0\}$  and  $\mathcal{C} \subseteq D'_1$ , it follows that for some  $i \in \{0, 1, \dots, n\}$ ,  $\{w_{i-1}, w_i\} <^p \{w_i, w_{i+1}\}$  (the indices are taken mod  $n+1$ ). It follows from the definition of  $t_1 = (D_0, U, A^<)$  that

$$A_{w_i}^< [\{w_{i-}(g) \mid g \text{ is a } w_{i+1}w_i \text{-arc}\}]$$

is a subpath of the subpath of  $A_{w_i}^<$  between the vertices  $w_{i-}(f_{i-1})$  and  $w_i^+$  and since  $f_{i-1} \in F(\mathcal{C}) \subseteq FD'_1$  it follows  $w_{i-}(f_{i-1}) \notin S_{w_i}$  and then  $\{w_{i-}(g) \mid g \text{ is a } w_{i+1}w_i \text{-arc}\} \cap S_{w_i} = \emptyset$ . Since  $f_i \in \mathcal{C} \subseteq D'_1$  and

$$\{w_{i-}(g) \mid g \text{ is a } w_{i+1}w_i \text{-arc}\} \cap S_{w_i} = \emptyset,$$

there exists some  $w_{i+1}w_i$ -arc in  $D_0$  which is also in  $D_0(S)$  and since  $D_1$  is an induced subdigraph of  $D_0$ , it follows that  $f_i \in F(\text{Sym } D_1)$  contradicting  $f_i \in F(D'_1)$ .

A digraph  $D$  is said to satisfy the *k-Meyniel's condition* if each odd directed cycle of  $D$  has at least  $k$  diagonals and we write  $D$  satisfies  $M(k)$ .

Let  $D_0$  be a digraph, we will denote by  $D_0^{(k)}$  the multidigraph obtained from  $D_0$  by adding to each symmetrical arc the multiplicity  $k$ . ■

**Lemma 3.1.** *If  $D_0$  is a digraph such that every odd directed cycle has a symmetrical arc and  $t_1 = (D_0^{(k)}, V(D_0), A^<, \gamma)$ , then  $\tau_1(t_1)$  satisfies  $M(k)$ .*

**Proof.** Let  $\mathcal{C}$  be an odd directed cycle contained in  $\tau_1(t_1)$ ; since  $\bigcup_{f \in FA_u^<} \gamma_u^f$  is a directed path of even length, we have that  $\mathcal{C}'$  is the digraph obtained from  $\mathcal{C}$  by identifying  $\bigcup_{f \in FA_u^<} \gamma_u^f$  with  $u$  for each  $u \in V(D_0)$ ;  $\mathcal{C}'$  is an odd directed cycle in  $D_0^{(k)}$  and clearly  $\mathcal{C} \cong t_1(\mathcal{C}', V(\mathcal{C}'), A^</V(\mathcal{C}'), \gamma')$ , where  $\gamma'$  is the restriction of  $\gamma$  to  $\bigcup_{u \in V(\mathcal{C}')} \bigcup_{f \in FA_u^<} \gamma_u^f$  and Definition 2.3 implies that each pseudodiagonal of  $\mathcal{C}'$  is a diagonal of  $\mathcal{C}$ . ■

As a direct consequence of Theorems 3.1, 2.5 and Lemma 3.1 we obtain.

**Theorem 3.2.** *If  $D_0$  is a KP-digraph (resp. CKI-digraph) such that every odd directed cycle has a symmetrical arc and  $t_1 = (D_0^{(k)}, V(D_0), A^<, \gamma)$ , then  $\tau_1(t_1)$  is a KP-digraph (resp. CKI-digraph) which satisfies  $M(k)$ .*

**Corollary 3.1.** *For each natural number  $k$ , there exists some KP-digraph (resp. CKI-digraph)  $D_k$  which satisfies the  $k$ -Meyniel's condition.*

**Proof.** Define the digraph  $C = \overrightarrow{C}_n(j_1, \dots, j_k)$  by  $V(C) = \{0, 1, \dots, n-1\}$ ,  $F(C) = \{uv \mid v - u \equiv j_s \pmod{n} \text{ for } s = 1, \dots, k\}$  and denote  $D_0 = \overrightarrow{C}_n(1, \pm 2, \dots, \pm r)$  for an even natural number  $n \not\equiv 0 \pmod{r+1}$ . In [6] it was proved that  $D_0$  is a CKI-digraph; so it follows from Theorems 3.1, 2.3 and Lemma 3.1 that  $\tau_1(t_1)$  is a CKI-digraph which satisfies  $M(k)$ . ■

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