

## DEGREE SEQUENCES OF DIGRAPHS WITH HIGHLY IRREGULAR PROPERTY

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### Abstract

A digraph such that for each its vertex, vertices of the out-neighbourhood have different in-degrees and vertices of the in-neighbourhood have different out-degrees, will be called an HI-digraph. In this paper, we give a characterization of sequences of pairs of out- and in-degrees of HI-digraphs.

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The results presented here are the continuation of studies concerning degree sequences of graphs with highly irregular property understood in such a way that the vertices of the neighbourhood of each vertex have different degrees.

For directed graphs irregularity of this type can be defined in various ways, because each vertex  $v$  have out-degree  $\deg^+(v)$ , in-degree  $\deg^-(v)$  and out-neighbourhood  $N^+(v)$ , in-neighbourhood  $N^-(v)$ . For example, in [1] authors consider digraphs  $G$  such that for every  $v \in V(G)$  the following implication is true:

$$(u, w \in N^+(v) \text{ and } u \neq w) \Rightarrow \deg^+(u) \neq \deg^+(w).$$

In this paper, some problems are investigated concerning the existence of such graphs with special properties. In particular, the class of ditrees is examined.

We are interested in the characterization of degree sequences of digraphs which have the property of irregularity explained above. As far as this is concerned, we found digraphs described by below definition especially interesting.

**Definition 1.** A digraph  $G$  will be called an *HI-digraph*, if for each vertex  $v \in V(G)$  the following condition holds:

$$(1) \quad \begin{cases} (u, w \in N^+(v) \text{ and } u \neq w) \Rightarrow \deg^-(u) \neq \deg^-(w), \\ (u, w \in N^-(v) \text{ and } u \neq w) \Rightarrow \deg^+(u) \neq \deg^+(w). \end{cases}$$

We assume that the digraphs considered here have at least one arc.

The purpose of our considerations is to get necessary and sufficient conditions for a sequence of pairs of non-negative integers to be the sequence of pairs of semi-degrees of an HI-digraph.

Let  $d^+ = (d_1^+, d_2^+, \dots, d_n^+)$  and  $d^- = (d_1^-, d_2^-, \dots, d_n^-)$  be sequences of non-negative integers. We say that the sequence of pairs

$$(d_1^+, d_1^-), (d_2^+, d_2^-), \dots, (d_n^+, d_n^-)$$

is *HI-digraphic* if there exists an HI-digraph  $G$  with the vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and such that  $\deg^+(v_j) = d_j^+$ ,  $\deg^-(v_j) = d_j^-$ , for  $j = 1, 2, \dots, n$ . Then  $G$  will be called an *HI-realization* of the sequence

$$(d_1^+, d_1^-), (d_2^+, d_2^-), \dots, (d_n^+, d_n^-).$$

**Proposition 1.** *If a sequence  $(d_j^+, d_j^-)$ ,  $j = 1, 2, \dots, n$ , is HI-digraphic, then for every permutation  $\phi$  of the set  $\{1, 2, \dots, n\}$  the sequence  $(d_j^+, d_{\phi(j)}^-)$ ,  $j = 1, 2, \dots, n$ , is HI-digraphic.*

**Proof.** Let  $G = (V, E)$ , where  $V = (v_1, v_2, \dots, v_n)$ , be an HI-realization of the sequence  $(d_j^+, d_j^-)$ ,  $j=1, 2, \dots, n$ . We define a new digraph  $G' = (V, E')$  such that

$$(v_i, v_j) \in E' \Leftrightarrow (v_i, v_{\phi(j)}) \in E.$$

Note that:

- (a)  $v_r \in N_{G'}^+(v_i) \Leftrightarrow v_{\phi(r)} \in N_G^+(v_i)$ ,
- (b)  $N_{G'}^-(v_j) = N_G^-(v_{\phi(j)})$ .

It gives:

- (c)  $\deg_{G'}^+(v_j) = \deg_G^+(v_j) = d_j^+$ ,
- (d)  $\deg_{G'}^-(v_j) = \deg_G^-(v_{\phi(j)}) = d_{\phi(j)}^-$ .

Now we show that  $G'$  is an HI-digraph. Let  $v_r$  and  $v_s$  be two vertices of out-neighbourhood of a vertex  $v_i$  in  $G'$ . Then, by (a),  $v_{\phi(r)}$  and  $v_{\phi(s)}$  are vertices of out-neighbourhood of a vertex  $v_i$  in  $G$ . Vertices  $v_{\phi(r)}$  and  $v_{\phi(s)}$  have different in-degrees in  $G$ . Then, by (d),  $v_r$  and  $v_s$  have different in-degrees in  $G'$ . Similarly, by (b) and (c), in the digraph  $G'$  vertices of in-neighbourhood have different out-degrees. ■

**Remark 1.** From Proposition 1 it follows that for characterization of HI-digraphic sequences of pairs  $(d_j^+, d_j^-)$ ,  $j = 1, 2, \dots, n$  it is sufficient to restrict our considerations to the case when both sequences

$$d^+ = (d_1^+, d_2^+, \dots, d_n^+) \quad \text{and} \quad d^- = (d_1^-, d_2^-, \dots, d_n^-)$$

are non-increasing. Then we say that the pair  $(d^+, d^-)$  is HI-digraphic.

If  $G$  is a digraph, then the maximum out-degree in  $G$  and the maximum in-degree in  $G$  will be denoted by  $\Delta^+(G)$  and  $\Delta^-(G)$ , respectively.

**Proposition 2.** *If  $G$  is an HI-digraph, then  $\Delta^+(G) = \Delta^-(G)$ .*

**Proof.** Let  $v$  and  $u$  be vertices of  $G$  for which  $\deg^+(v) = \Delta^+(G)$  and  $\deg^-(u) = \Delta^-(G)$ . Note that:

- (1) the set  $\{\deg^-(w) : w \in N^+(v)\}$  has exactly  $\deg^+(v)$  elements (by HI-graphicity of  $G$ ),
- (2) if  $w \in N^+(v)$ , then  $\deg^-(w) \neq 0$ ,
- (3) if some set  $\Omega$  consists of  $\omega$  positive integers, then  $\omega \leq \max \Omega$ .

By (1) we have

$$\Delta^+(G) = \deg^+(v) = |\{\deg^-(w) : w \in N^+(v)\}|,$$

and, by (2), (3),

$$|\{\deg^-(w) : w \in N^+(v)\}| \leq \max\{\deg^-(w) : w \in N^+(v)\} \leq \Delta^-(G).$$

So,

$$\Delta^+(G) \leq \Delta^-(G).$$

Similarly, we obtain

$$\Delta^-(G) = \deg^-(u) \leq \max\{\deg^+(w) : w \in N^-(u)\} \leq \Delta^+(G). \quad \blacksquare$$

Proposition 2 permits to put:

$$\Delta(G) = \Delta^+(G) = \Delta^-(G)$$

for an HI-digraph  $G$ .

Let

$$S^+(G) = \{s : s \neq 0 \text{ and } s = \deg^+(w) \text{ for some } w \in V(G)\},$$

$$S^-(G) = \{s : s \neq 0 \text{ and } s = \deg^-(w) \text{ for some } w \in V(G)\}.$$

**Proposition 3.** *If  $G$  is an HI-digraph, then  $|S^+(G)| = \Delta(G) = |S^-(G)|$ .*

**Proof.** Let  $v$  be a vertex of  $G$  such that  $\deg^+(v) = \Delta^+(G)$ . Note that

- (1)  $\Delta^+(G) = |\{\deg^-(w) : w \in N^+(v)\}|$ ,
- (2)  $\{\deg^-(w) : w \in N^+(v)\} \subseteq S^-(G)$  and
- (3)  $|S^-(G)| \leq \max S^-(G) \leq \Delta^-(G)$ .

Applying, in turn, (1), (2) and (3) we obtain

$$\Delta^+(G) \leq |S^-(G)| \leq \Delta^-(G).$$

Thus, by Proposition 2, we have  $|S^-(G)| = \Delta(G)$ . Similarly, we prove that  $\Delta(G) = |S^+(G)|$ . ■

**Theorem 1.** *If a pair  $(d^+, d^-)$  of non-increasing sequences of non-negative integers has an HI-realization  $G$  with  $\Delta(G) = m$ , then*

$$(2) \left\{ \begin{array}{l} d^+ = (\underbrace{m, \dots, m}_{n_m}, \underbrace{m-1, \dots, m-1}_{n_{m-1}}, \dots, \underbrace{i, \dots, i}_{n_i}, \dots, \underbrace{1, \dots, 1}_{n_1}, \underbrace{0, \dots, 0}_{n_0}) \\ d^- = (\underbrace{m, \dots, m}_{k_m}, \underbrace{m-1, \dots, m-1}_{k_{m-1}}, \dots, \underbrace{i, \dots, i}_{k_i}, \dots, \underbrace{1, \dots, 1}_{k_1}, \underbrace{0, \dots, 0}_{k_0}) \\ \text{where } n_0, k_0 \text{ are non-negative integers and } m, n_i, k_i \in \mathbb{N}, \\ n_i \geq k_m, k_i \geq n_m \text{ for } i = 1, 2, \dots, m, \\ \sum_{i=0}^m n_i = \sum_{j=0}^m k_j \text{ and } \sum_{i=1}^m i \cdot n_i = \sum_{j=1}^m j \cdot k_j. \end{array} \right.$$

**Proof.** Let  $n_i$  ( $k_i$ ) denote the number of vertices of  $G$  with out-degree (in-degree)  $i$ ,  $i = 0, 1, \dots, m$ . By Proposition 3, the numbers  $n_i$  and  $k_i$  are positive for  $i = 1, 2, \dots, m$ . From Definition 1 it follows that every vertex with out-degree  $m$  is joined with exactly one vertex of in-degree  $i$  for each  $i = 1, 2, \dots, m$  and every vertex with in-degree  $i$  is joined with at most one vertex of out-degree  $m$ . Then  $k_i \geq n_m$  for  $i = 1, 2, \dots, m$ . By the similar argumentation we obtain:  $n_i \geq k_m$  for  $i = 1, 2, \dots, m$ . The last equations of (2) hold by graphicity of  $(d^+, d^-)$ . ■

According to Theorem 1 says that the degree sequences of HI-digraphs with maximum degree  $m$  (out and in) have the analogous form as degree sequences of undirected HI-graphs (characterization of HI-graphic sequences has been presented in [4]). Namely, in each of them all integers of the set  $\{1, 2, \dots, m\}$  appear, moreover the number of elements equal to  $m$  in both sequences is the same. Observe that in (2) the number  $n_0$  or  $k_0$  can be equal to zero. By a remark below, we can restrict our consideration to one case when  $n_0$  and  $k_0$  are both positive.

**Remark 2.** The consideration of HI-graphicity of the pair  $(d^+, d^-)$  of the form (2) can be restricted to the case in which  $n_0 \neq 0$  and  $k_0 \neq 0$ .

**Proof.** Let  $(d^+, d^-)$  be a pair of sequences of the form (2) and let  $(t^+, t^-)$  be the pair obtained by adding to both sequences  $d^+$  and  $d^-$  one element equal to 0. Note that adding to HI-digraph  $G$ , being a realization of  $(d^+, d^-)$ , a new isolated vertex we obtain an HI-realization  $T$  of  $(t^+, t^-)$ . Conversely, deleting from an HI-digraph  $T$ , being a realization of  $(t^+, t^-)$ , one isolated vertex we obtain an HI-realization  $G$  of  $(d^+, d^-)$ . ■

The case  $n_0 \neq 0$  and  $k_0 \neq 0$  everywhere extorts the assumption that HI-digraph has at least one isolated vertex, however in this case the notations are far uniform. The class of HI-digraphs with at least one isolated vertex we will denote by  $\text{HI}_0$ .

Let sequences  $a = (a_1, a_2, \dots, a_n)$ ,  $b = (b_1, b_2, \dots, b_k)$  and a matrix  $[c_{ij}]_{n \times k}$  have elements being non-negative integers. Let  $\mathcal{M} = (X \cup Y, \mu)$ , where  $X = \{x_1, x_2, \dots, x_n\}$ ,  $Y = \{y_1, y_2, \dots, y_k\}$  and  $\mu : X \times Y \rightarrow N \cup \{0\}$  be a bipartite oriented multigraph. We will say that the multigraph  $\mathcal{M}$  is a realization of the pair  $(a, b)$  with bounds  $c_{ij}$  if for  $i = 1, 2, \dots, n, j = 1, 2, \dots, k$  :  $\deg^+(x_i) = a_i$ ,  $\deg^-(y_j) = b_j$  and  $\mu(x_i, y_j) \leq c_{ij}$ .

**Lemma 1.** Let  $(d^+, d^-)$  be a pair of sequences of the form (2), where  $n_0 \neq 0$  and  $k_0 \neq 0$ , and let  $b^+ = (b_0^+, b_1^+, \dots, b_m^+)$ ,  $b^- = (b_0^-, b_1^-, \dots, b_m^-)$  be sequences

such that  $b_i^+ = i \cdot n_i$ ,  $b_j^- = j \cdot k_j$ ,  $i, j = 0, 1, \dots, m$ . If the pair  $(d^+, d^-)$  is  $\text{HI}_0$ -digraphic, then there exists a bipartite oriented multigraph  $\mathcal{M} = (X \cup Y, \mu)$ , where  $\mu : X \times Y \rightarrow N \cup \{0\}$ , which realizes the pair  $(b^+, b^-)$  with bounds  $c_{ij} = \min\{n_i, k_j\}$ ,  $i, j = 0, 1, \dots, m$ .

**Proof.** Let  $G = (V, E)$  be an  $\text{HI}_0$ -digraph which realizes the pair  $(d^+, d^-)$ . We put:

$$\begin{aligned} X_i &= \{u \in V : \deg^+(u) = i\}, Y_j = \{u \in V : \deg^-(u) = j\}, \\ \mu(X_i, Y_j) &= |\{(u, v) \in E : u \in X_i, v \in Y_j\}| \text{ for } i, j = 0, 1, \dots, m, \\ X &= \{X_0, X_1, \dots, X_m\}, Y = \{Y_0, Y_1, \dots, Y_m\}. \end{aligned}$$

It is not difficult to check that the multigraph  $\mathcal{M} = (X \cup Y, \mu)$  has the required properties.  $\blacksquare$

The multigraph  $\mathcal{M}$  described in the proof of Lemma 1 will be denoted by  $\mathcal{M}_G$ .

Further we will prove the converse of Lemma 1. For this we use results of paper [3] adopting them for our aims.

Let  $G = (V, E)$  be a digraph in which  $V = \{v_1, v_2, \dots, v_n\}$  and let  $\{d_0^+, d_1^+, \dots, d_r^+\}$ ,  $\{d_0^-, d_1^-, \dots, d_s^-\}$  be the sets of out- and in-degrees in  $G$ , respectively. We assume that  $d_0^+ < d_1^+ < \dots < d_r^+$  and  $d_0^- < d_1^- < \dots < d_s^-$ . For  $i = 0, 1, \dots, r$  and  $j = 0, 1, \dots, s$  we put:

$$V_i^+ = \{v \in V : \deg_G^+(v) = d_i^+\},$$

$$V_j^- = \{v \in V : \deg_G^-(v) = d_j^-\},$$

Then for each vertex  $v$  of  $G$  we define column-vectors  $\vec{v}^+$  and  $\vec{v}^-$  in the following way:

$$\vec{v}^+ = (t_0^+(v), t_1^+(v), \dots, t_s^+(v))^T, \text{ where } t_j^+(v) \text{ is equal to the number of arcs from the vertex } v \text{ to vertices of the set } V_j^-,$$

$$\vec{v}^- = (t_0^-(v), t_1^-(v), \dots, t_r^-(v))^T, \text{ where } t_i^-(v) \text{ is equal to the number of arcs from vertices of the set } V_i^+ \text{ to the vertex } v.$$

In this way, with the digraph  $G$ , we can associate the pair  $(M_G^+, M_G^-)$  of matrices, where

$$M_G^+ = [\vec{v}_1^+, \vec{v}_2^+, \dots, \vec{v}_n^+] \quad \text{and}$$

$$M_G^- = [\vec{v}_1^-, \vec{v}_2^-, \dots, \vec{v}_n^-].$$

Note that  $M_G^+$  is a  $((s+1) \times n)$ -matrix and  $M_G^-$  is a  $((r+1) \times n)$ -matrix (in [3] they have been called out- and in-distribution matrix, respectively).

By  $\widetilde{M}_G^+$  we denote the matrix obtained from  $M_G^+$  by a reordering of columns as follows:

$$\vec{v}_r^+ \text{ precedes } \vec{v}_s^+ \text{ in } \widetilde{M}_G^+ \\ \Updownarrow$$

(the sum of members of  $\vec{v}_r^+$  is less than the sum of members of  $\vec{v}_s^+$ ) or  
(these sums are equal and  $r < s$ ).

Similarly, we define the matrix  $\widetilde{M}_G^-$ . By  $\widetilde{M}_i^+$ ,  $i = 0, 1, \dots, r$ , we denote the submatrix of  $\widetilde{M}_G^+$  which contains all columns with the sum of members equal to  $d_i^+$ . Analogously, we define  $\widetilde{M}_j^-$ ,  $j = 0, 1, \dots, s$ . Then we can identify the matrices  $\widetilde{M}_G^+$  and  $\widetilde{M}_G^-$  with the sequences

$$(\widetilde{M}_0^+, \widetilde{M}_1^+, \dots, \widetilde{M}_r^+) \quad \text{and} \quad (\widetilde{M}_0^-, \widetilde{M}_1^-, \dots, \widetilde{M}_s^-),$$

respectively.

We say that a pair  $(A, B)$  of sequences of matrices is  $\text{HI}_0$ -digraphic if there exists a digraph  $G$  of the class  $\text{HI}_0$  such that  $A = \widetilde{M}_G^+$  and  $B = \widetilde{M}_G^-$ . Then the digraph  $G$  will be called an  $\text{HI}_0$ -realization of  $(A, B)$ .

It is easy to check that the following proposition is true.

**Proposition 4.** *If  $G$  is  $\text{HI}_0$ -digraph with  $\Delta(G) = m$ , then  $\widetilde{M}_G^+ = (A_0, A_1, \dots, A_m)$  and  $\widetilde{M}_G^- = (B_0, B_1, \dots, B_m)$ , where for  $i = 0, 1, \dots, m$  the following conditions hold:*

- (i) *each of matrices  $A_i$ ,  $B_i$  has  $m + 1$  rows,*
- (ii) *all matrices  $A_i$ ,  $B_i$  have elements equal to 0 or 1, only,*
- (iii) *each column of  $A_i$  ( $B_i$ ) has exactly  $i$  elements equal to 1.*

In [3] the definition of a demi-bipartite graph is given. Namely, a *demi-bipartite graph* (*d.b-graph*) is a triple  $(X, Y, E)$ , where  $X, Y$  are non-empty sets and  $E \subseteq X \times Y$  (note that the sets  $X$  and  $Y$  can have an element in common).

A d.b-graph  $(X, Y, E)$ , where  $X = \{x_1, x_2, \dots, x_n\}$  and  $Y = \{y_1, y_2, \dots, y_k\}$  is called a *realization* of a pair  $(a, b) = ((a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_k))$  of sequences of non-negative integers, if for any  $i = 1, 2, \dots, n$ ,  $j = 1, 2, \dots, k$ :

$$\deg^+(x_i) = a_i, \deg^-(y_j) = b_j, \quad \text{and}$$

$$\deg^-(x_i) = 0, \deg^+(y_j) = 0 \quad \text{for } x_i, y_j \notin X \cap Y.$$

From the criterion of d.b-graphicity of  $(a, b)$ , (see [3] Lemma 4), the following follows immediately

**Proposition 5.** *If  $a$  and  $b$  are 0-1-sequences then  $(a, b)$  is realizable by a d.b-graph if and only if both sequences  $a$  and  $b$  have the same number of members equal to 1.*

**Lemma 2.** *Let  $m$  be a positive integer and let  $A = (A_0, A_1, \dots, A_m)$ ,  $B = (B_0, B_1, \dots, B_m)$  be sequences of matrices, where  $A_i = [a_{pq}^i]$ ,  $p = 0, 1, \dots, m$  and  $q = 1, 2, \dots, n_i$ ;  $B_j = [b_{rs}^j]$ ,  $r = 0, 1, \dots, m$  and  $s = 1, 2, \dots, k_j$ . Moreover let  $\sum_{i=0}^m n_i = \sum_{j=0}^m k_j$  and for  $A$  and  $B$  the conditions (i) – (iii) of Proposition 4 hold. Then  $(A, B)$  is  $\text{HI}_0$ -digrahic if and only if*

- (iv) *for every  $i, j = 0, 1, \dots, m$  the  $j$ -th row of  $A_i$  and  $i$ -th row of  $B_j$  have the same number of members equal to 1.*

**Proof.** ( $\Rightarrow$ ) Let  $G = (V, E)$  be an  $\text{HI}_0$ -realization of the pair  $(A, B)$ . We put:  $V_i^+ = \{v \in V : \deg^+(v) = i\}$ ,  $V_j^- = \{v \in V : \deg^-(v) = j\}$ ,  $E_{ij} = \{(u, v) \in E : u \in V_i^+ \text{ and } v \in V_j^-\}$ . We consider the d.b-graphs  $D_{ij} = (V_i^+, V_j^-, E_{ij})$ . Note that each digraph  $D_{ij}$  realizes the pair of sequences

$$((a_{j1}^i, a_{j2}^i, \dots, a_{jn_i}^i), (b_{i1}^j, b_{i2}^j, \dots, b_{ik_j}^j)).$$

Thus, by Proposition 5, we have (iv).

( $\Leftarrow$ ) Let  $V$  be an arbitrary  $n$ -element set and let  $\{X_i\}_{i \in \{0, 1, \dots, m\}}$ ,  $\{Y_j\}_{j \in \{0, 1, \dots, m\}}$  be two partitions of the set  $V$  such that  $|X_i| = n_i$  and  $|Y_j| = k_j$  and  $X_0 \cap Y_0 \neq \emptyset$ . For  $i, j = 0, 1, \dots, m$  we put:  $X_i = \{x_1^i, x_2^i, \dots, x_{n_i}^i\}$ ,  $Y_j = \{y_1^j, y_2^j, \dots, y_{k_j}^j\}$ . From Proposition 5 it follows that for every  $i, j \in \{0, 1, \dots, m\}$  there exists a d.b-graph  $H_{ij} = (X_i, Y_j, F_{ij})$  which is a realization of the pair

$$((a_{j1}^i, a_{j2}^i, \dots, a_{jn_i}^i), (b_{i1}^j, b_{i2}^j, \dots, b_{ik_j}^j)).$$

This means that if  $(x_q^i, y_s^j) \in F_{ij}$  then  $a_{jq}^i = 1$  and  $b_{is}^j = 1$ . Moreover, if  $i_1 \neq i_2$  ( $j_1 \neq j_2$ ) then  $F_{i_1 j} \cap F_{i_2 j} = \emptyset$  ( $F_{i_1 j_1} \cap F_{i_1 j_2} = \emptyset$ ).

We consider the digraph  $G = (V, F)$ , where  $F = \bigcup_{i,j \in \{0, 1, \dots, m\}} F_{ij}$ . We will prove that  $X_i = \{v \in V : \deg^+(v) = i\}$ ,  $Y_j = \{v \in V : \deg^-(v) = j\}$ ,  $G \in \text{HI}_0$  and  $G$  realizes the pair  $(A, B)$ .

Let  $v \in X_i$ . Then  $v = x_q^i$  for some  $q \in \{1, 2, \dots, n_i\}$ . If  $(x_q^i, y) \in F$ , then  $(x_q^i, y) \in F_{ij}$  for some  $j \in \{0, 1, \dots, m\}$ . Thus,  $a_{jq}^i = 1$ . Since the sets  $F_{ij}$  are mutually disjoint, the number of arcs from  $x_q^i$  in  $F$  is equal to the number of elements equal to 1 in  $q$ -th column of the matrix  $A_i$ . By (iii),  $\deg^+(v) = i$ . In a similar way we obtain: if  $v \in Y_j$  then  $\deg^-(v) = j$ .



Let  $v \in V$  and let  $u, w \in N^+(v)$ ,  $u \neq w$ . Then  $(v, u) \in F_{ij_1}$  and  $(v, w) \in F_{ij_2}$  for some  $i, j_1, j_2 \in \{0, 1, \dots, m\}$ . Since d.b-graphs  $H_{ij_1}$  and  $H_{ij_2}$  realize the pairs of 0-1-sequences, so  $j_1 \neq j_2$ . Thus  $\deg^-(u) \neq \deg^-(w)$ . Obviously, vertices belonging to the set  $X_0 \cap Y_0$  are isolated.

It is not difficult to check that  $\widetilde{M}^+(G) = A$  and  $\widetilde{M}^-(G) = B$ . ■

Now we give the converse of Lemma 1.

**Lemma 3.** *Let  $(d^+, d^-)$  be a pair of sequences of the form (2), where  $n_0 \neq 0$  and  $k_0 \neq 0$ , and let*

$$b^+ = (b_0^+, b_1^+, \dots, b_m^+), \quad b^- = (b_0^-, b_1^-, \dots, b_m^-)$$

*be sequences such that  $b_i^+ = i \cdot n_i$ ,  $b_j^- = j \cdot k_j$ ,  $i, j = 0, 1, \dots, m$ . If there exists a bipartite oriented multigraph  $\mathcal{M} = (X \cup Y, \mu)$ , where  $\mu : X \times Y \rightarrow N \cup \{0\}$ , which realizes the pair  $(b^+, b^-)$  with bounds  $c_{ij} = \min\{n_i, k_j\}$ ,  $i, j = 0, 1, \dots, m$ , then the pair  $(d^+, d^-)$  is  $\text{HI}_0$ -digraphic.*

**Proof.** Let  $\mathcal{M} = (X, Y, \mu)$ , where

$$X = \{x_0, x_1, \dots, x_m\} \quad \text{and} \quad Y = \{y_0, y_1, \dots, y_m\},$$

be a realization of  $(b^+, b^-)$  with the bounds  $c_{ij}$ . By Lemma 2 it is sufficient to prove that there exists a pair  $(A, B)$  of sequences of matrices for which conditions (i) – (iv) hold. We will give a method of construction of the sequences  $A = (A_0, A_1, \dots, A_m)$  and  $B = (B_0, B_1, \dots, B_m)$ .

For each  $i = 0, 1, \dots, m$  we construct the matrix  $A_i = [a_{pq}^i]_{(m+1) \times n_i}$  in the following way. As  $A_0$  we put 0-matrix of the order  $(m+1) \times n_0$ . For  $i \geq 1$ , let  $\alpha^{+i} = (\alpha_1^{+i}, \alpha_2^{+i}, \dots, \alpha_h^{+i})$  be the sequence of the form:

$$\underbrace{1, \dots, 1}_{\mu_{i1}}, \dots, \underbrace{j, \dots, j}_{\mu_{ij}}, \dots, \underbrace{m, \dots, m}_{\mu_{im}},$$

where

$$\mu_{ij} = \mu(x_i, y_j), \quad j = 1, 2, \dots, m$$

and let  $\beta^{+i} = (\beta_1^{+i}, \beta_2^{+i}, \dots, \beta_k^{+i})$  be the sequence which is formed by repetition,  $i$  times, of the sequence  $(1, 2, \dots, n_i)$ . Note that the lengths  $h$  and  $k$  of sequences  $\alpha^{+i}$  and  $\beta^{+i}$ , respectively, are equal because  $h = \sum_{j=1}^m \mu_{ij} = b_i^+$  and  $k = i \cdot n_i = b_i^+$ . Let  $I^{+i} = \{(\alpha_t^{+i}, \beta_t^{+i}); t = 1, 2, \dots, k\}$ .

Note that  $I^{+i} \subseteq \{(p, q) : p = 1, 2, \dots, m, q = 1, 2, \dots, n_i\}$ . Then we define the 0-1-matrix  $A_i = [a_{pq}^i]$  putting:  $a_{0q}^i = 0$  for every  $q \in \{1, 2, \dots, n_i\}$  and  $a_{pq}^i = 1$  if and only if  $(p, q) \in I^{+i}$  for  $p \geq 1$ .

Similarly, for each  $j = 0, 1, \dots, m$  we construct the matrix  $B_j = [b_{rs}^j]_{(m+1) \times k_j}$ . We define  $B_0$  as the 0-matrix of the order  $(m+1) \times k_0$ . For  $j \geq 1$  we consider the sequence  $\alpha^{-j} = (\alpha_1^{-j}, \alpha_2^{-j}, \dots, \alpha_h^{-j})$  of the form:

$$\underbrace{1, \dots, 1}_{\mu_{1j}}, \dots, \underbrace{i, \dots, i}_{\mu_{ij}}, \dots, \underbrace{m, \dots, m}_{\mu_{mj}},$$

where

$$\mu_{ij} = \mu(x_i, y_j), \quad i = 1, 2, \dots, m$$

and the sequence  $\beta^{-j} = (\beta_1^{-j}, \beta_2^{-j}, \dots, \beta_k^{-j})$  formed by repetition,  $j$  times, of the sequence  $(1, 2, \dots, k_j)$ .

Obviously, the sequences  $\alpha^{-j}$  and  $\beta^{-j}$  have the same number of elements. Let  $I^{-j} = \{(\alpha_t^{-j}, \beta_t^{-j}) : t = 1, 2, \dots, k\}$ . Then  $B_j = [b_{rs}^j]$ , where  $b_{0s}^j = 0$  for every  $s \in \{1, 2, \dots, k_j\}$  and  $b_{rs}^j = 1$  if and only if  $(r, s) \in I^{-j}$  for  $r \geq 1$ .

Obviously, for the matrices  $A_i$  and  $B_j$  conditions (i) and (ii) hold.

Let  $q \in \{1, 2, \dots, n_i\}$  and  $p \in \{1, 2, \dots, m\}$ . Note that the number  $q$  appears exactly  $i$  times in the sequences  $\beta^{+i}$  ( $q = \beta_t^{+i}$  for  $t \in \{q, q+n_i, \dots, q+(i-1)n_i\}$ ) however, the number  $p$  is repeated  $\mu_{ip}$  times in the sequence  $\alpha^{+i}$ , where  $\mu_{ip} \leq c_{ip} = \min\{n_i, k_p\} \leq n_i$ . So, the set  $I^{+i}$  has exactly  $i$  pairs with second element equal to  $q$ . Then  $\sum_{p=1}^{m+1} a_{pq}^i = i$ . By similar argumentation we have  $\sum_{r=1}^{m+1} b_{rs}^j = j$  for  $s = 1, 2, \dots, k_j$ . Thus the condition (iii) holds. To prove condition (iv) it is sufficient to note that:

- the number of elements equal to 1 in  $i$ -th row of the matrix  $B_j$  is equal to the number of elements equal to  $i$  in the sequence  $\alpha^{-j}$  and
- the number of elements equal to 1 in  $j$ -th row of the matrix  $A_i$  is equal to the number of elements equal to  $j$  in the sequence  $\alpha^{+i}$ .

By definition of sequences  $\alpha^{-j}$  and  $\alpha^{+i}$  these numbers are equal to  $\mu_{ij}$ . ■

**Example.** An example which demonstrates the method of constructing the  $\text{HI}_0$ -digraph with the help of results presented in this paper will be presented.

Let  $d^+ = (3, 3, 2, 2, 2, 2, 1, 1, 1, 0, 0)$  and  $d^- = (3, 3, 2, 2, 2, 2, 1, 1, 1, 1, 0)$ . Accordingly to Lemma 3 we form the sequences  $b^+ = (0, 3, 8, 6)$  and

$b^- = (0, 5, 6, 6)$ . The multigraph  $\mathcal{M}$  in Figure 1 (the numbers about arcs denote the multiplicity of these arcs) realizes the pair  $(b^+, b^-)$  with bounds  $c_{ij}$ ,  $i, j = 0, 1, 2, 3$ , given by the matrix

$$[c_{ij}] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 1 & 4 & 3 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix}.$$

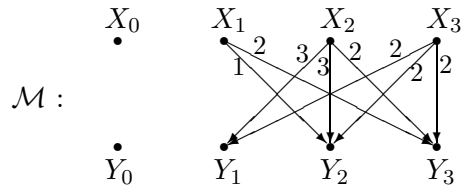


Figure 1

According to the procedure given in the proof of Lemma 3 the following auxiliary sequences  $\alpha^{+i}$ ,  $\beta^{+i}$  and  $\alpha^{-i}$ ,  $\beta^{-i}$  are constructed:

$$\begin{aligned} \alpha^{+1} &= (2, 3, 3) & \alpha^{+2} &= (1, 1, 1, 2, 2, 2, 3, 3) & \alpha^{+3} &= (1, 1, 2, 2, 3, 3) \\ \beta^{+1} &= (1, 2, 3) & \beta^{+2} &= (1, 2, 3, 4, 1, 2, 3, 4) & \beta^{+3} &= (1, 2, 1, 2, 1, 2) \\ \alpha^{-1} &= (2, 2, 2, 3, 3) & \alpha^{-2} &= (1, 2, 2, 2, 3, 3) & \alpha^{-3} &= (1, 1, 2, 2, 3, 3) \\ \beta^{-1} &= (1, 2, 3, 4, 5) & \beta^{-2} &= (1, 2, 3, 1, 2, 3) & \beta^{-3} &= (1, 2, 1, 2, 1, 2) \end{aligned}.$$

Then we obtain the following pair  $(A, B)$  of sequences of matrices:

$$\begin{aligned} A &= \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \right) \\ B &= \left( \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \right). \end{aligned}$$

These matrices satisfy conditions (i) – (iv). To construct an HI-digraph which realizes the pair  $(A, B)$  we apply the method given in the proof of Theorem 3.

Let  $V = \{v_1, v_2, \dots, v_{11}\}$  and let

$X_0 = \{v_{11}, v_{10}\}$ ,  $X_1 = \{v_9, v_8, v_7\}$ ,  $X_2 = \{v_6, v_5, v_4, v_3\}$ ,  $X_3 = \{v_2, v_1\}$ ,  
 $Y_0 = \{v_{11}\}$ ,  $Y_1 = \{v_{10}, v_9, v_8, v_7, v_6\}$ ,  $Y_2 = \{v_5, v_4, v_3\}$ ,  $Y_3 = \{v_1, v_2\}$ .

All d.b-graphs  $H_{ij}$  for  $i = 0$  or  $j = 0$  and  $H_{11}$  have no arcs. The remaining d.b-graphs are presented in Figure 2.

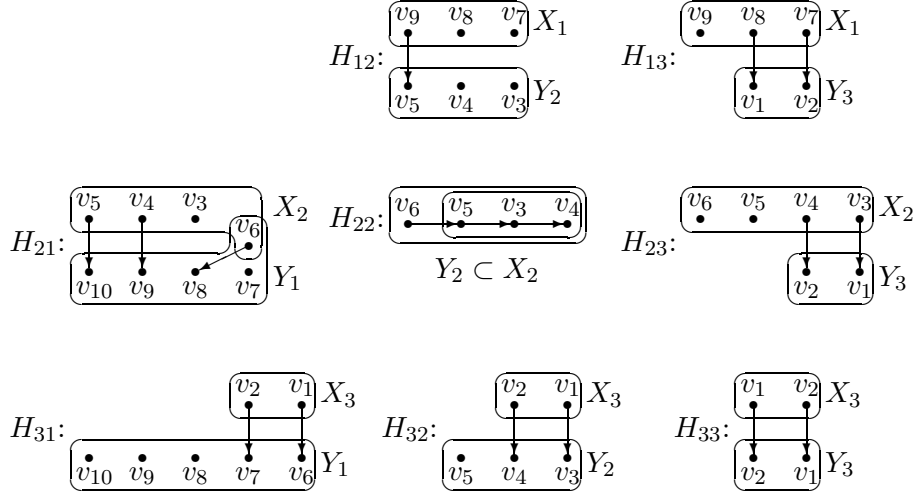


Figure 2

Then the digraph  $G$  (see Figure 3) which realizes the pair  $(d^+, d^-)$  is the union of the above d.b-graphs.

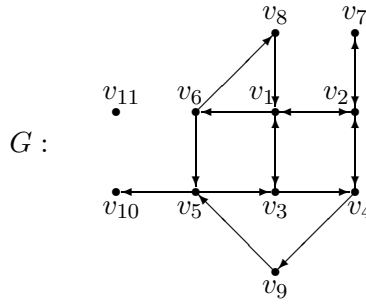


Figure 3

Finally, we give the main theorem of this paper which is a combinatorial characterization of the degree sequences of HI-digraphs. We use Hoffman's criterion ([2]) which is a characterization of the degree sequences of bipartite multigraphs with bounds for the multiplicity of edges. This criterion can be formulated as follows:

Let  $\mathbf{d} = (d_1, d_2, \dots, d_m)$  and  $\tilde{\mathbf{d}} = (d_{m+1}, d_{m+2}, \dots, d_{m+n})$  be sequences of non-negative integers such that  $\sum_{i=1}^m d_i = \sum_{j=m+1}^{m+n} d_j = s$  and let  $c_{ij} \geq 0$  for  $1 \leq i \leq m$ ,  $m+1 \leq j \leq m+n$  be given non-negative integers. There exists a bipartite multigraph which is a realization of the pair  $(\mathbf{d}, \tilde{\mathbf{d}})$  with the bounds  $c_{ij}$  if and only if, for every  $I \subseteq \{1, 2, \dots, m\}$ ,  $J \subseteq \{m+1, m+2, \dots, m+n\}$  we have

$$\sum_{i \in I, j \in J} c_{ij} \geq \sum_{i \in I} d_i + \sum_{j \in J} d_j - s$$

From Lemma 1, Lemma 3 and above Hoffman's criterion, in the face of Remark 2, it immediately follows:

**Theorem 2.** *A pair  $(d^+, d^-)$  of sequences of the form (2) is HI-digraphic if and only if for every  $I, J \subseteq \{0, 1, \dots, m\}$  the following inequality holds:*

$$(3) \quad \sum_{i \in I, j \in J} \min\{n_i, k_j\} \geq \sum_{i \in I} i \cdot n_i + \sum_{j \in J} j \cdot k_j - \sum_{i=0}^m i \cdot n_i.$$

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